Ecole BasMatI
Bases mathématiques pour l'instrumentation et le traitement du signal en astronomie

Nice - Porquerolles, 1 - 5 Juin 2015

# Representation of signals as series of orthogonal functions

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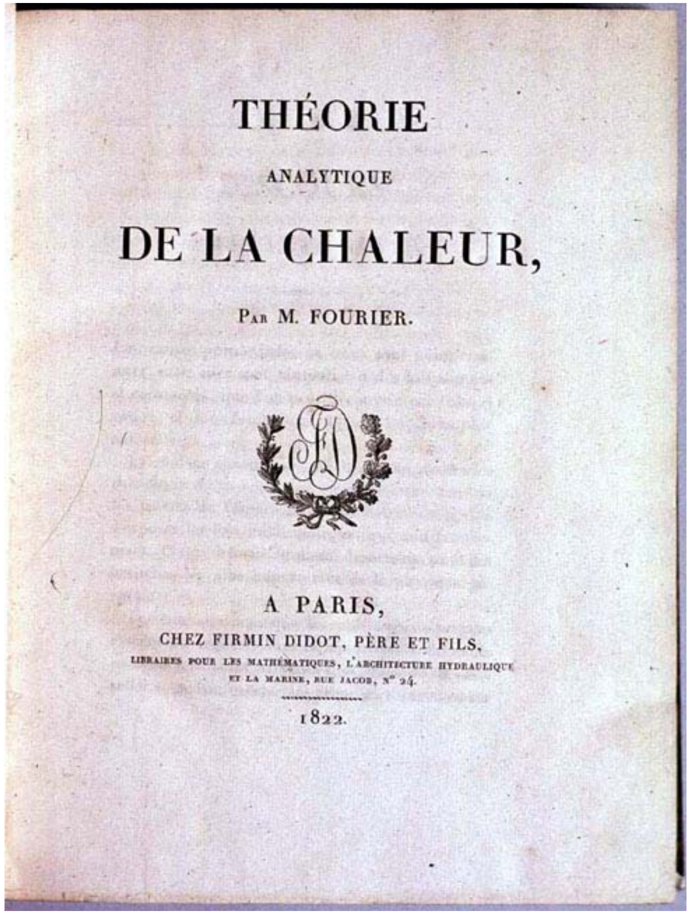
1. Fourier Series

2. Legendre polynomials

3. Spherical harmonics

4. Bessel functions

# Fourier Series



THÉORIE
ANALYTIQUE
DE LA CHALEUR,
PAR M. FOURIER.

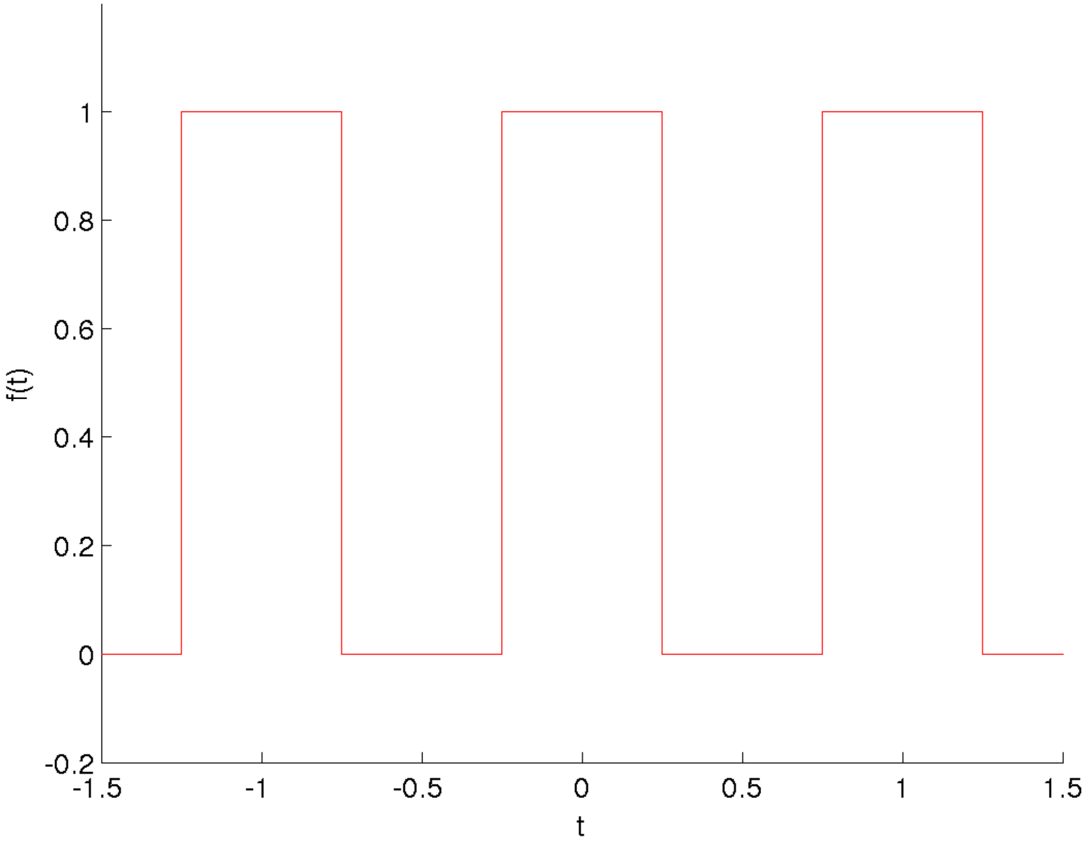
A PARIS,
CHEZ FIRMIN DIDOT, PÈRE ET FILS,
LIBRAIRES POUR LES MATHÉMATIQUES, L'ARCHITECTURE HYDRAULIQUE
ET LA MARINE, RUE JACOB, N° 24.
1822.

Théorie analytique de la chaleur, 1822

Solutions to the heat equation (diffusion PDE) as trigonometric series

# Fourier Series

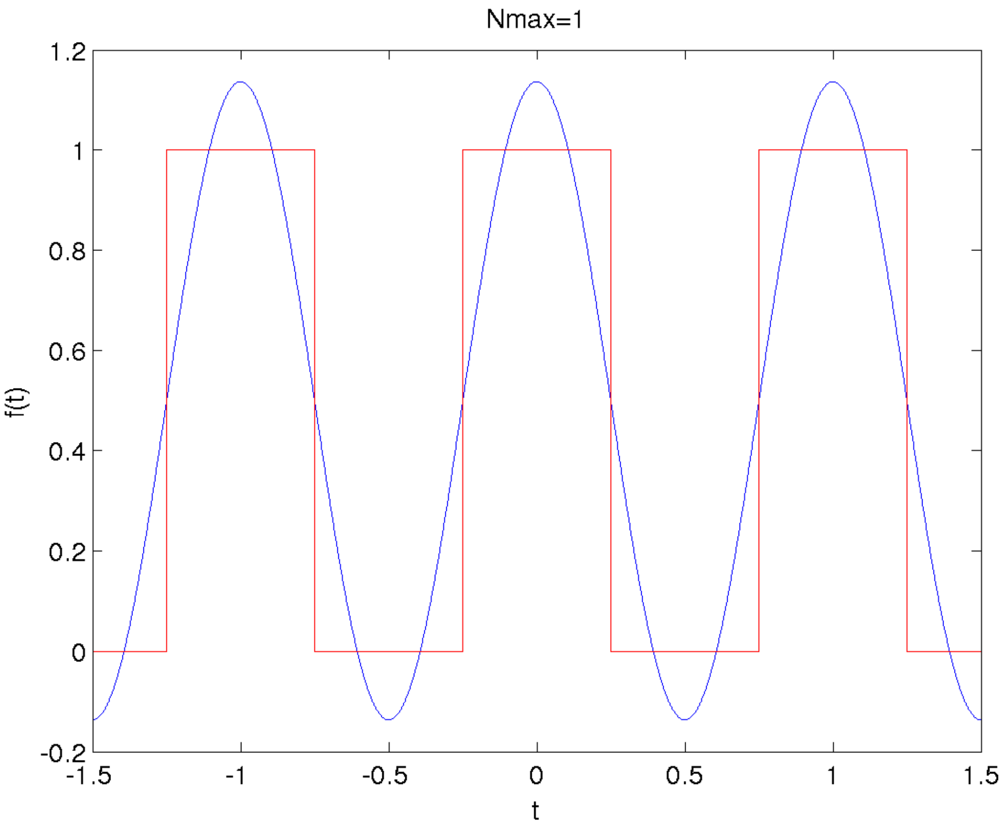
## Approching a periodic signal by a sum of trigonometric functions



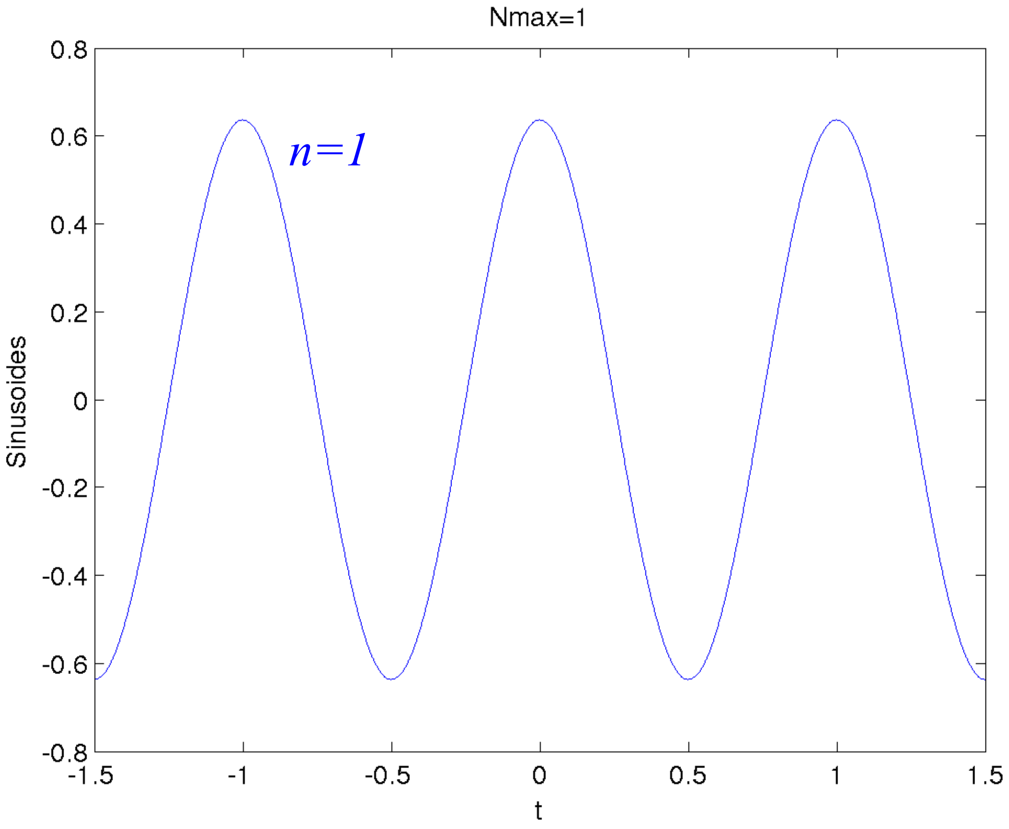
For a signal with period $T$

$$f(t)=\sum_{n=\underset{-N_{max}}{\cancel{-\infty}}}^{\overset{N_{max}}{\cancel{\infty}}} c_n \boxed{e^{2i\pi\frac{nt}{T}}}$$

frequency $n/T$

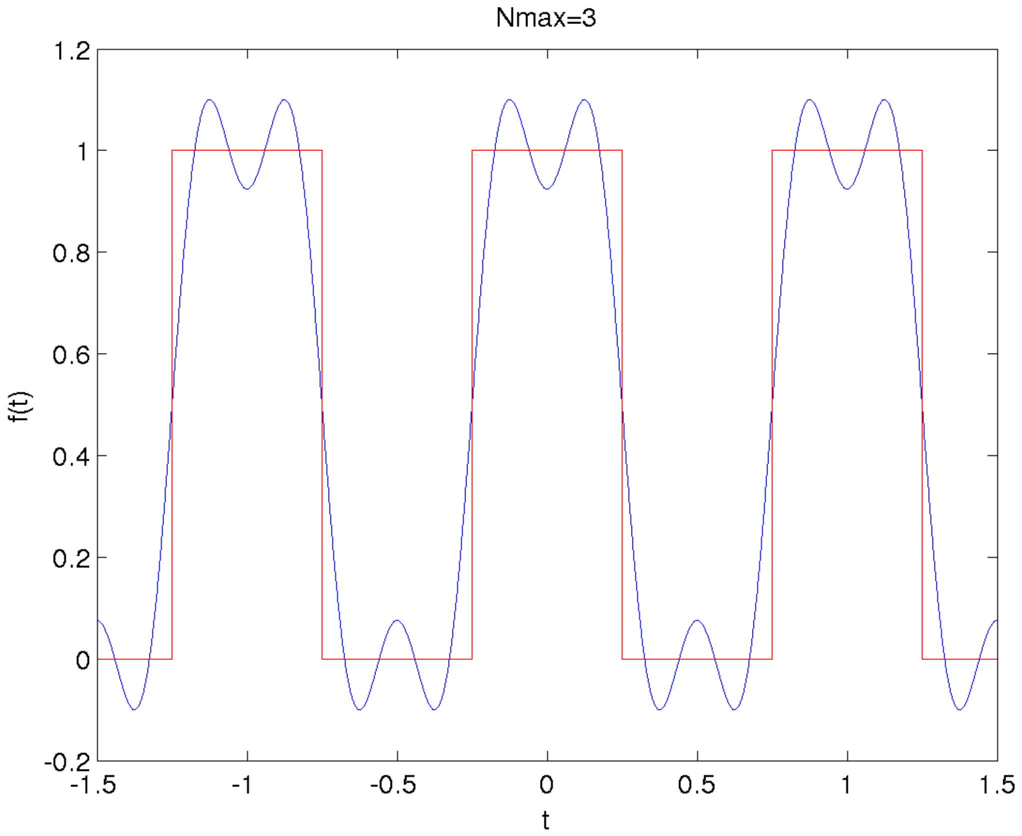


Partial Fourier series

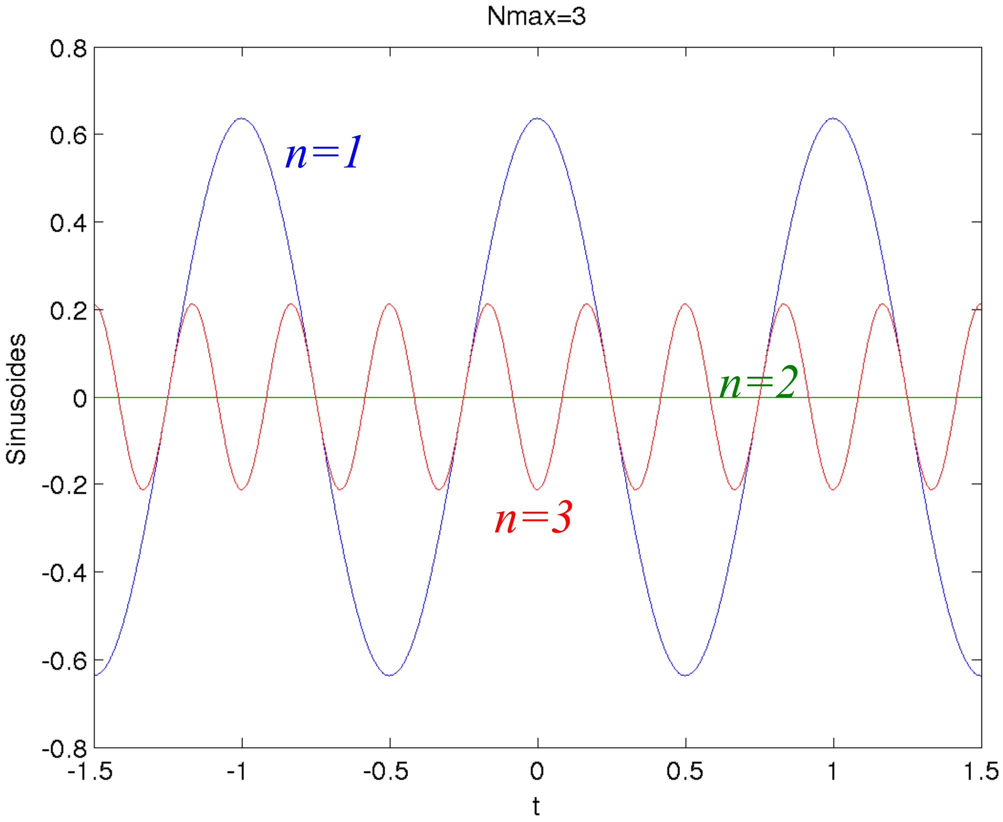


Individual terms

Frequencies : $0$ , $\pm 1/T$

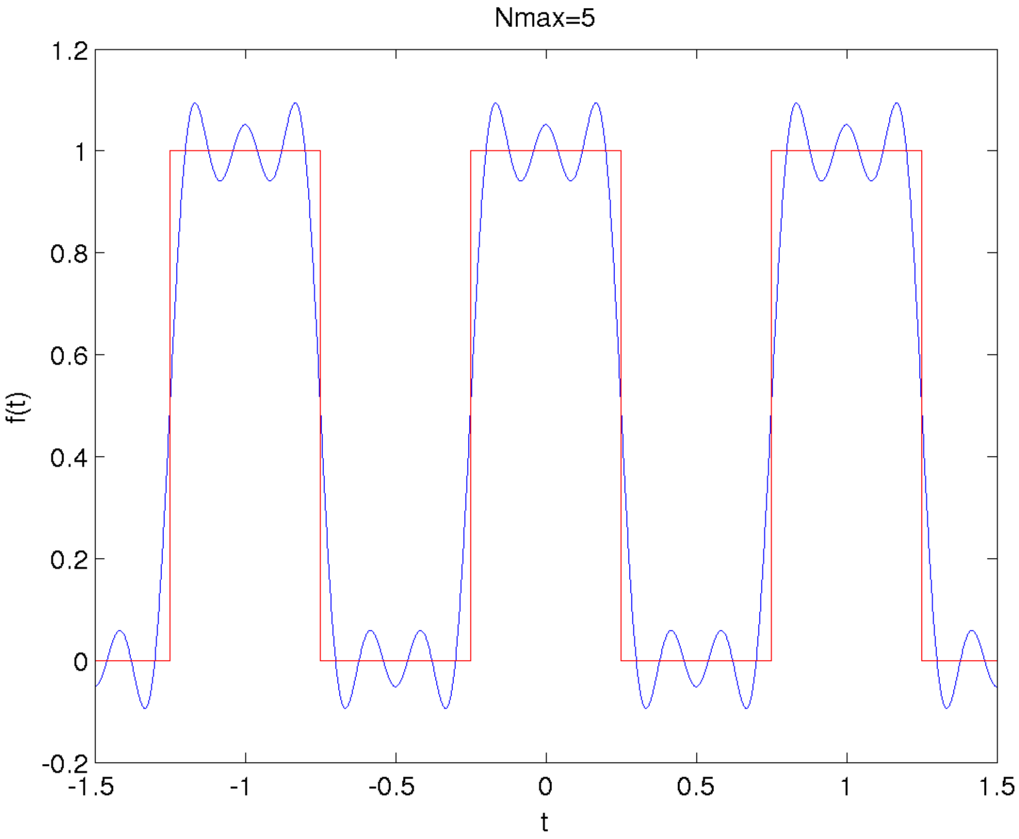


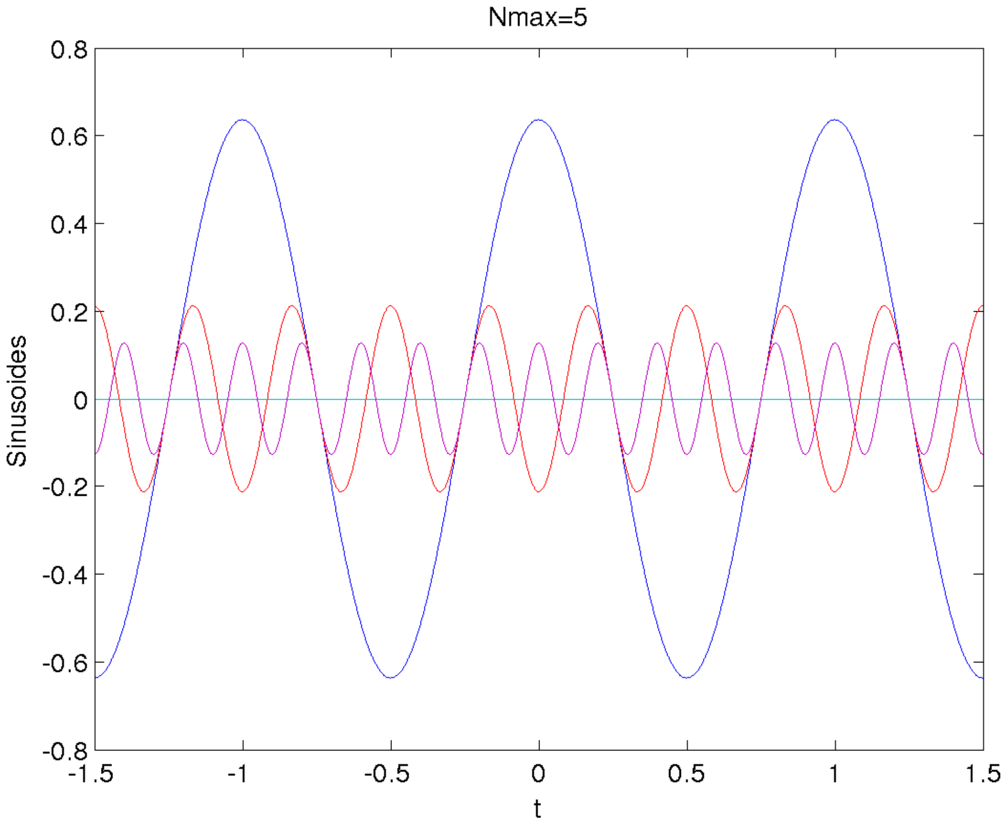
Partial Fourier series



Individual terms

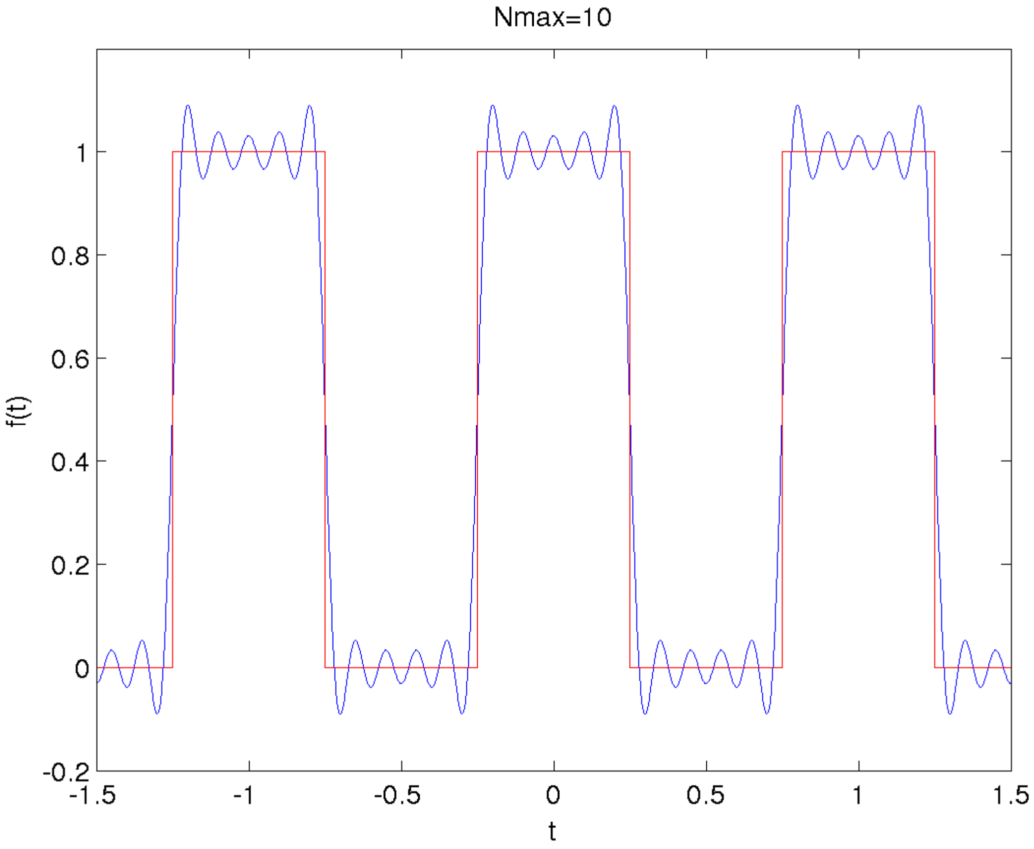
Frequencies : $0 , \pm 1/T , \pm 2/T, \pm 3/T$



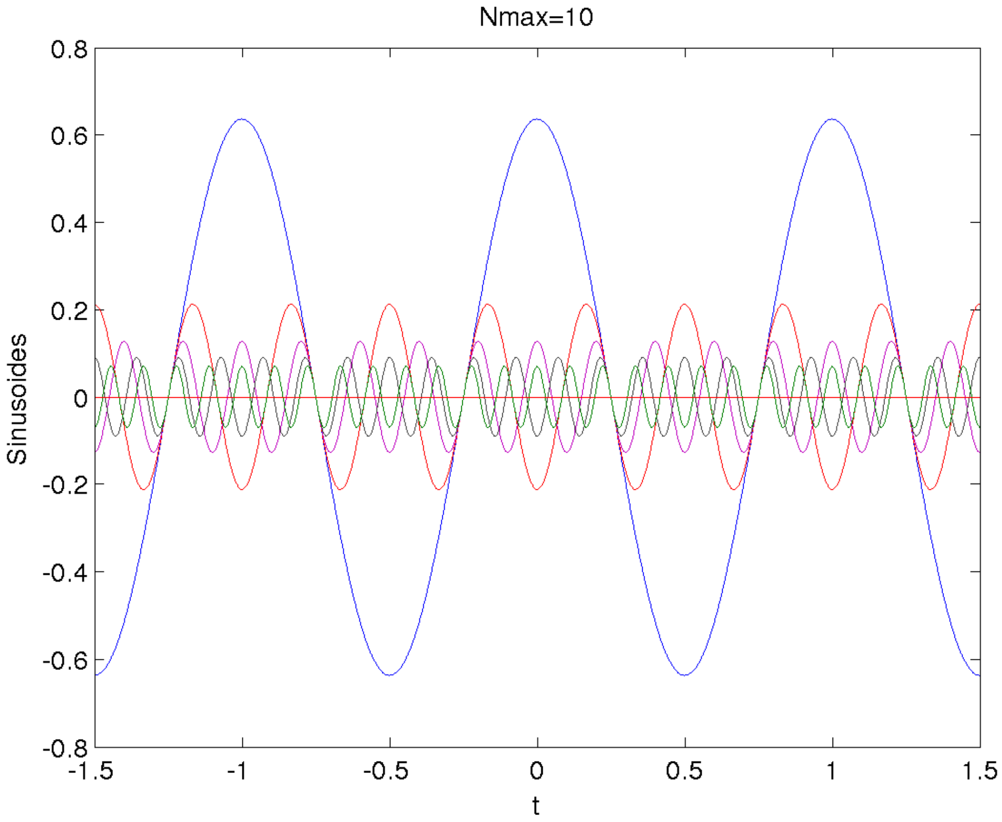


Partial Fourier series

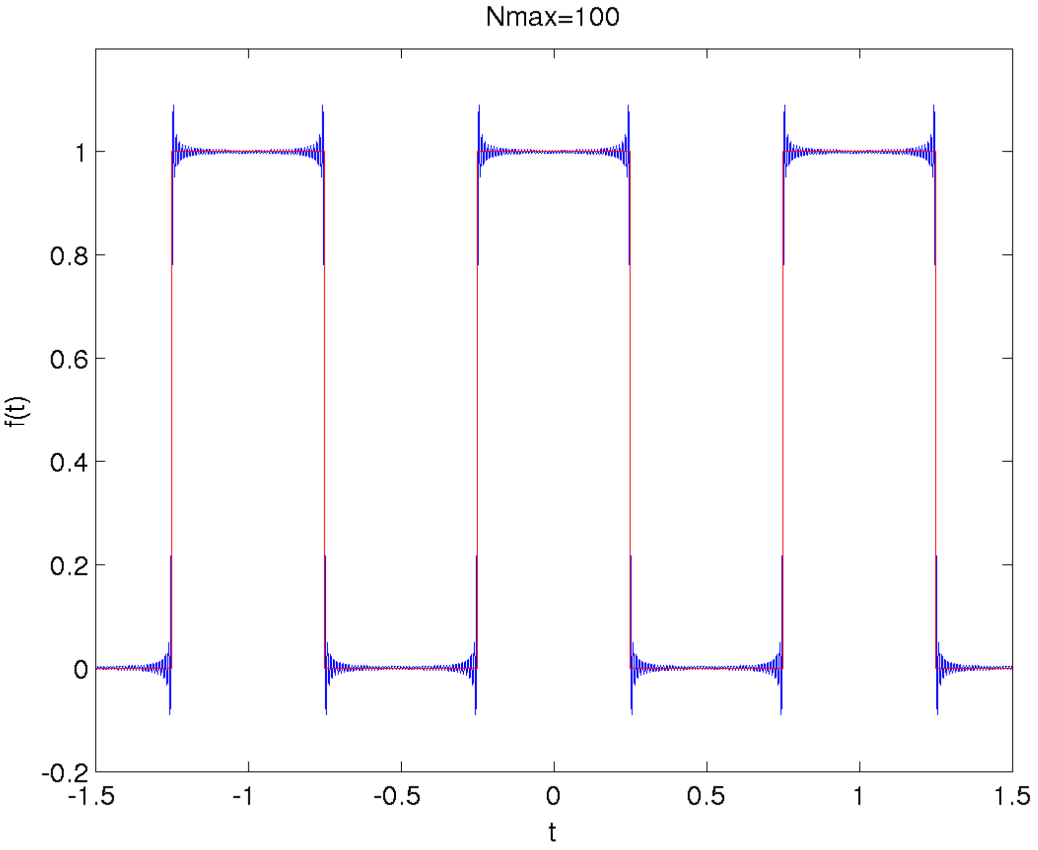
Individual terms



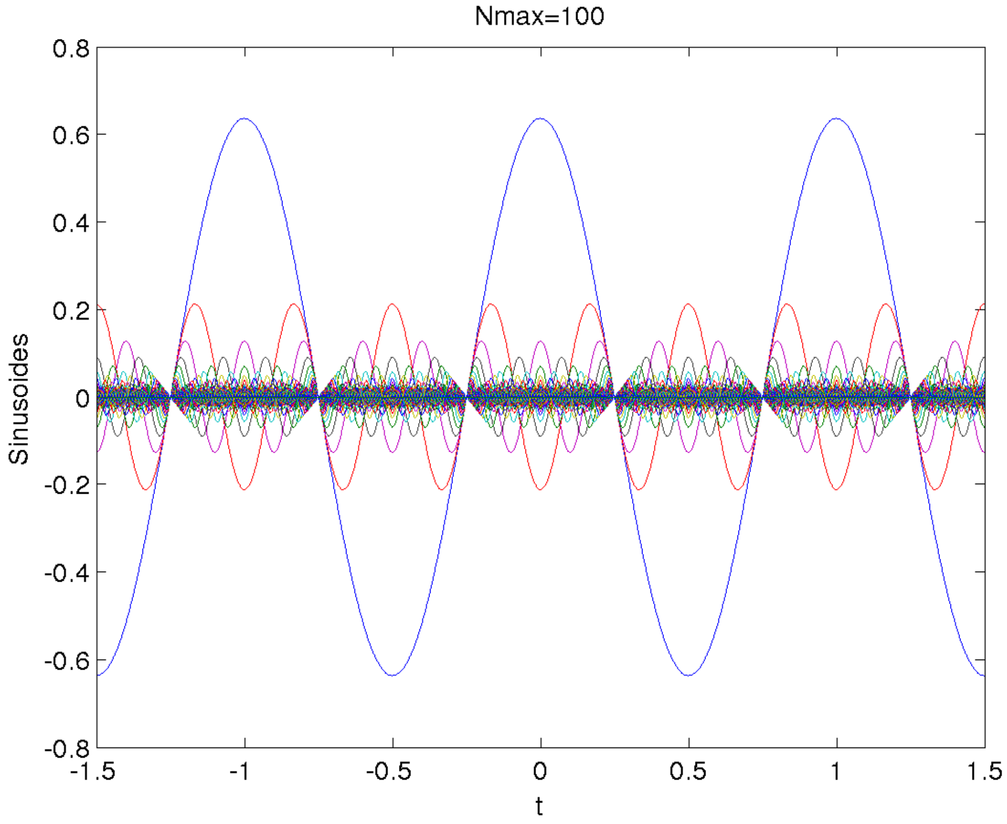
Partial Fourier series



Individual terms

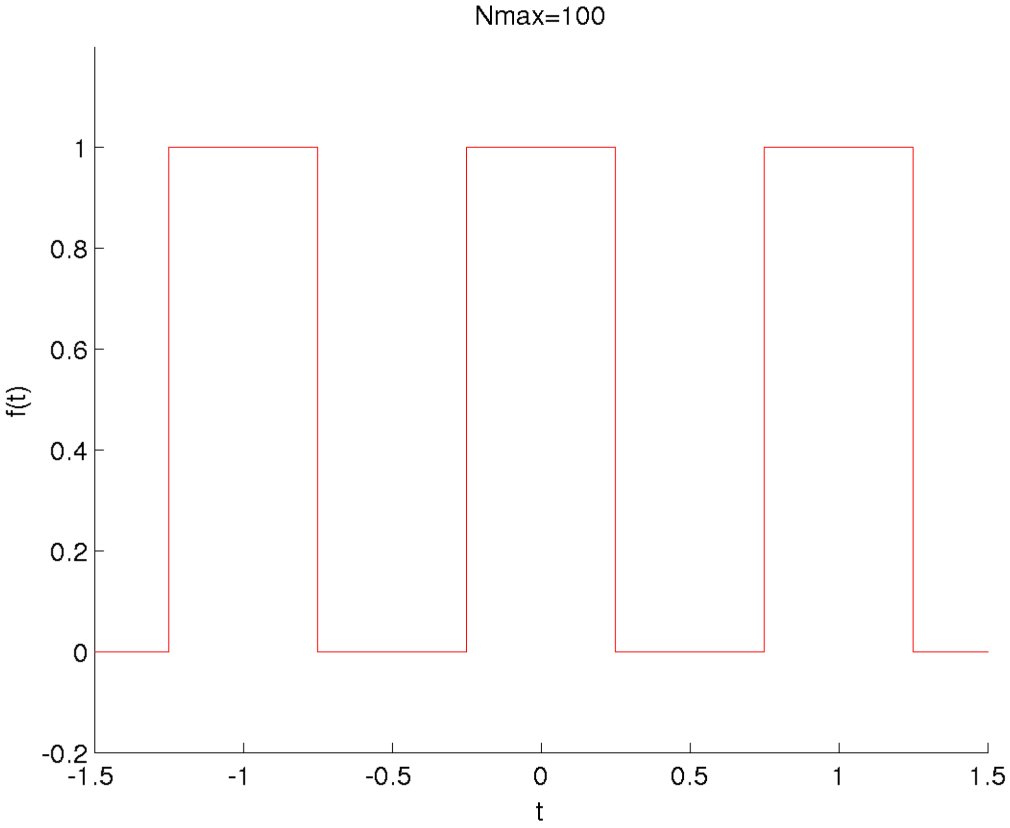


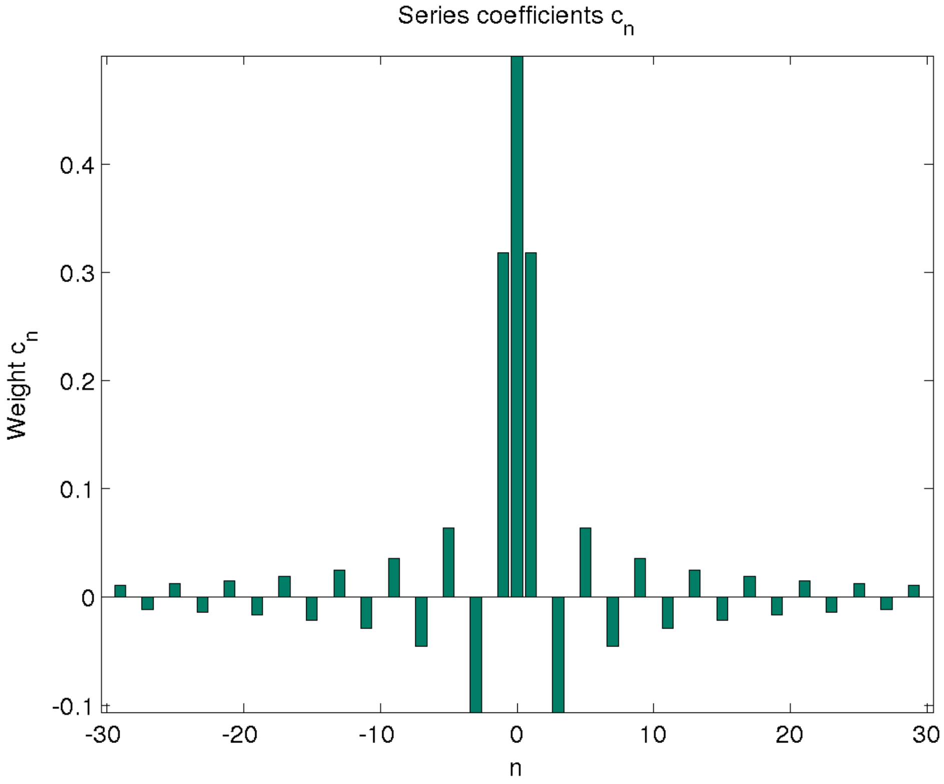
Partial Fourier series



Individual terms

# Two representations of the signal





Ensemble of discrete sampled values $\{f(t_k)\}$

Ensemble of Fourier coefficients $\{c_n\}$

# Fourier Series

period $T/n$

Complex form :
for a signal $f(t)$ of period $T$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2i\pi \frac{nt}{T}}$$

Even functions : sum of cosines

$$c_n = c_{-n}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{nt}{T}\right)$$

$$a_n = c_n + c_{-n} \qquad a_0 = c_0$$

Odd functions : sum of sines

$$c_n = -c_{-n}$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(2\pi \frac{nt}{T}\right)$$

$$b_n = i(c_n - c_{-n})$$

General case ; real form : $f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{nt}{T}\right) + b_n \sin\left(2\pi \frac{nt}{T}\right)$

# Calculating the coefficients $c_n$

$$\phi_n(t) = e^{2i\pi\frac{nt}{T}}$$

weighting function

« Key » relation :

$$\int_0^T \frac{1}{T}\,\phi_n(t)\,\overline{\phi_m(t)}\,dt = \left| \begin{array}{ll} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{array}\right.$$

Calculate the integral

$$\int_0^T \frac{1}{T}\,f(t)\,\overline{\phi_n(t)}\,dt$$

$$f(t) = \sum_{p=-\infty}^{\infty} c_p\phi_p(t)$$

$$= \int_0^T \frac{1}{T}\left[\sum_p c_p\phi_p(t)\right]\overline{\phi_n(t)}\,dt \qquad = \sum_p c_p \boxed{\int_0^T \frac{1}{T}\,\phi_p(t)\,\overline{\phi_n(t)}\,dt}$$

$$= c_n$$

uniform convergence

nul if $p \neq n$

Then :

$$\boxed{c_n = \int_0^T \frac{1}{T}\,f(t)\,\overline{\phi_n(t)}\,dt}$$

# Fourier series and scalar product

## Vectors (3D)

$\vec{u} = (u_1, u_2, u_3) \qquad \vec{v} = (v_1, v_2, v_3)$

Scalar product : $\vec{u}.\vec{v} = \sum_i u_i v_i$

*(bilinear, for u and v)*

2 vectors... 1 number

Norm : $||\vec{u}||^2 = \vec{u}.\vec{u}$

*(positive)*

Orthogonality : $\vec{u} \perp \vec{v} \iff \vec{u}.\vec{v} = 0$

Orthonormal base : $(\hat{x}, \hat{y}, \hat{z})$

$$\left[ \begin{array}{l} \hat{x}.\hat{y} = 0 \\ \hat{x}.\hat{x} = 1 \end{array} \right.$$

Decomposition :

$$\vec{u} = u_1\hat{x} + u_2\hat{y} + u_3\hat{z}$$

with $u_1 = \vec{u}.\hat{x}$

## Functions

$f(t), g(t)$

Scalar product : $(f, g) = \int_0^T \frac{1}{T} f(t)\, \overline{g(t)}\, dt$

(suited to Fourier series)

Norm : $(f, f) = \int_0^T \frac{1}{T} |f(t)|^2\, dt$

$f$ and $g$ are orthogonal iif $(f, g) = 0$

Orthonormal base : $\{\phi_n\}$

$$(\phi_n, \phi_m) = \left| \begin{array}{ll} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{array} \right.$$

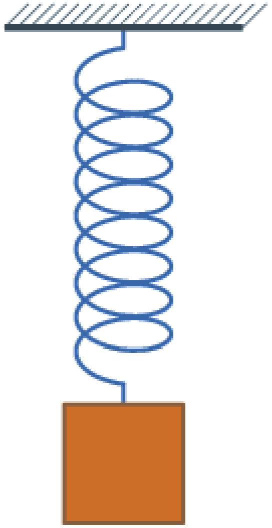
Decomposition :

$$f(t) = \sum_n c_n\, \phi_n(t)$$

= Fourier series

$c_n = (f, \phi_n)$

# Fourier series and Differential Equations



(trigonometric series were introduced from Fourier work on PDE heat equation)

Harmonic oscillator equation $f'' + \omega_n^2 f = 0$

with $\omega_n = \frac{2\pi n}{T}$

2 independent solutions are $\phi_n(t) = e^{2i\pi \frac{nt}{T}}$

and $\phi_{-n}(t) = e^{-2i\pi \frac{nt}{T}}$

Any solution is the superposition $f(t) = k_1 \phi_n(t) \ + \ k_2 \phi_{-n}(t)$

Fourier base functions are associated with harmonic differential equation

# Essential ideas on Fourier Series

- ☛ Periodic signal
- ☛ Can be expressed as a series of trigonometric base functions (exp, cos, sin)
- ☛ Base functions are orthonormal, with respect to an appropriate scalar product
- ☛ Coefs. of the series calculated as a scalar product between the signal and each base function
- ☛ Base functions are solutions of a differential Eq.

# Representation of signals as series of orthogonal functions

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3. Spherical harmonics
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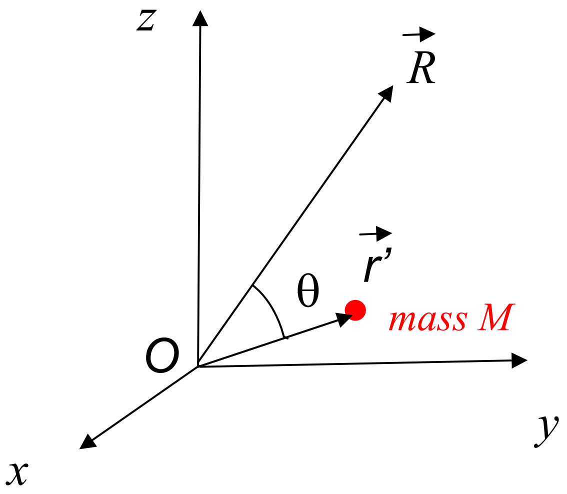
# Legendre Polynomials

Introduced by A.M. Legendre, 1784, « Recherches sur la figure des planètes », mém. de l'académie royale des sciences de Paris



Gravitational potential at $\vec{R}$

$$\Phi(\vec{R}) = \frac{GM}{|\vec{R} - \vec{r}'|}$$

with :

$$|\vec{R} - \vec{r}'| = R\sqrt{1 - 2\frac{r'}{R}\cos\theta + \left(\frac{r'}{R}\right)^2}$$

Introducing $r = \frac{r'}{R}$ and $x = \cos\theta$

$$\Phi(\vec{R}) \propto \boxed{[1 - 2rx + r^2]^{-\frac{1}{2}}}$$

Generating function of Legendre Polynomials

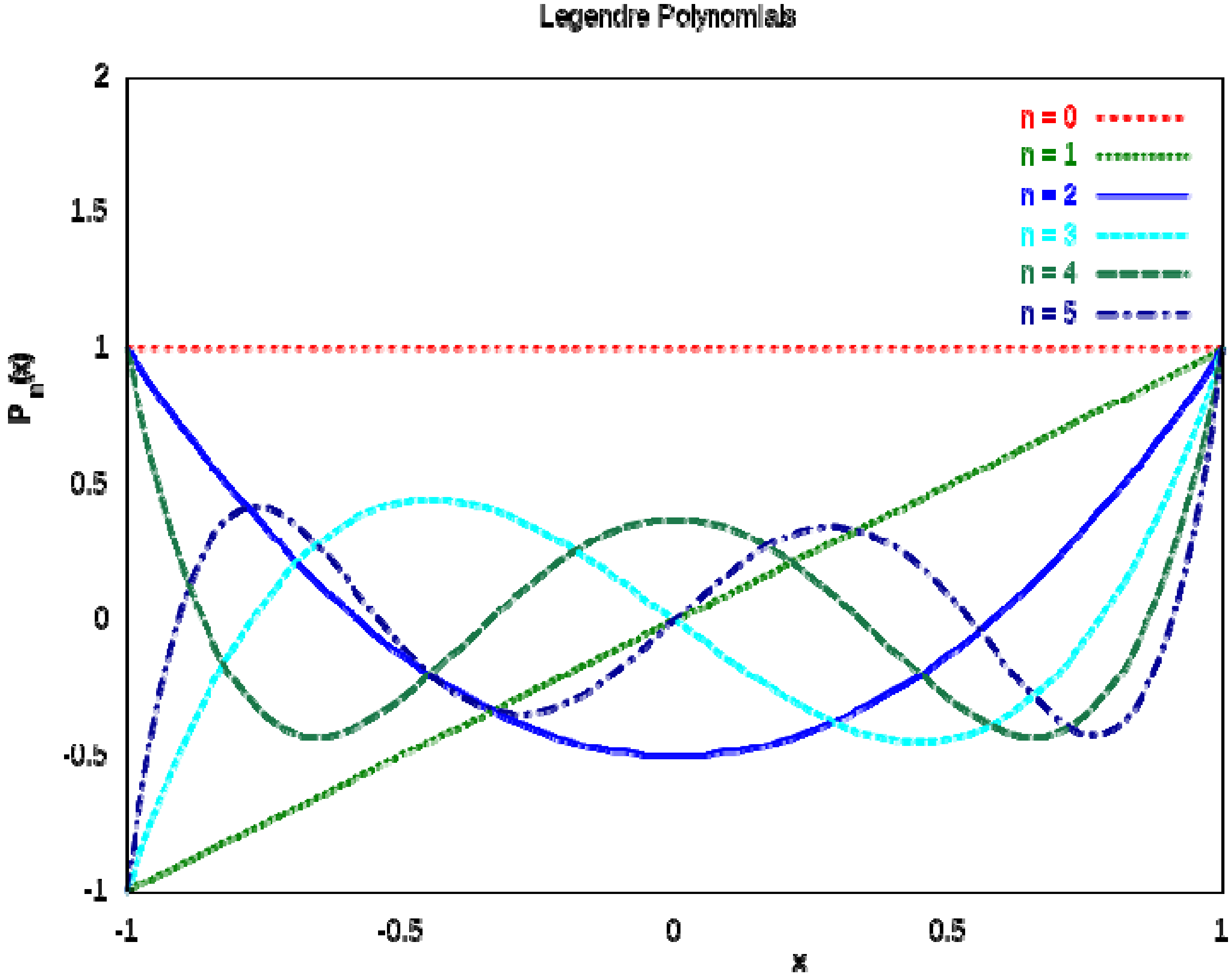
Taylor expansion at $r=0$ : $[1-2rx+r^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} r^n P_n(x)$ with $-1 \leq x \leq 1$

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$



Recurrence relation :

$$(n+1)P_{n+1}(x) =$$

$$(2n+1)xP_n(x) - nP_{n-1}(x)$$

Degree : $n$ Parity : $n$

$n$ distinct roots in the interval [1-,1]

# Fourier-Legendre series

Scalar product :
$$(f,g)=\int_{-1}^{1} f(x)\,g(x)\,dx$$

(suited to Legendre polynomials)

Orthogonality :

$$(P_n,P_m)=\int_{-1}^{1} P_n(x)\,P_m(x)\,dx=\frac{1}{n+\frac{1}{2}}\,\delta_{mn}=\left|\begin{array}{ll} 0 & \text{if } n\neq m \\ \frac{1}{n+1/2} & \text{if } n=m \end{array}\right.$$

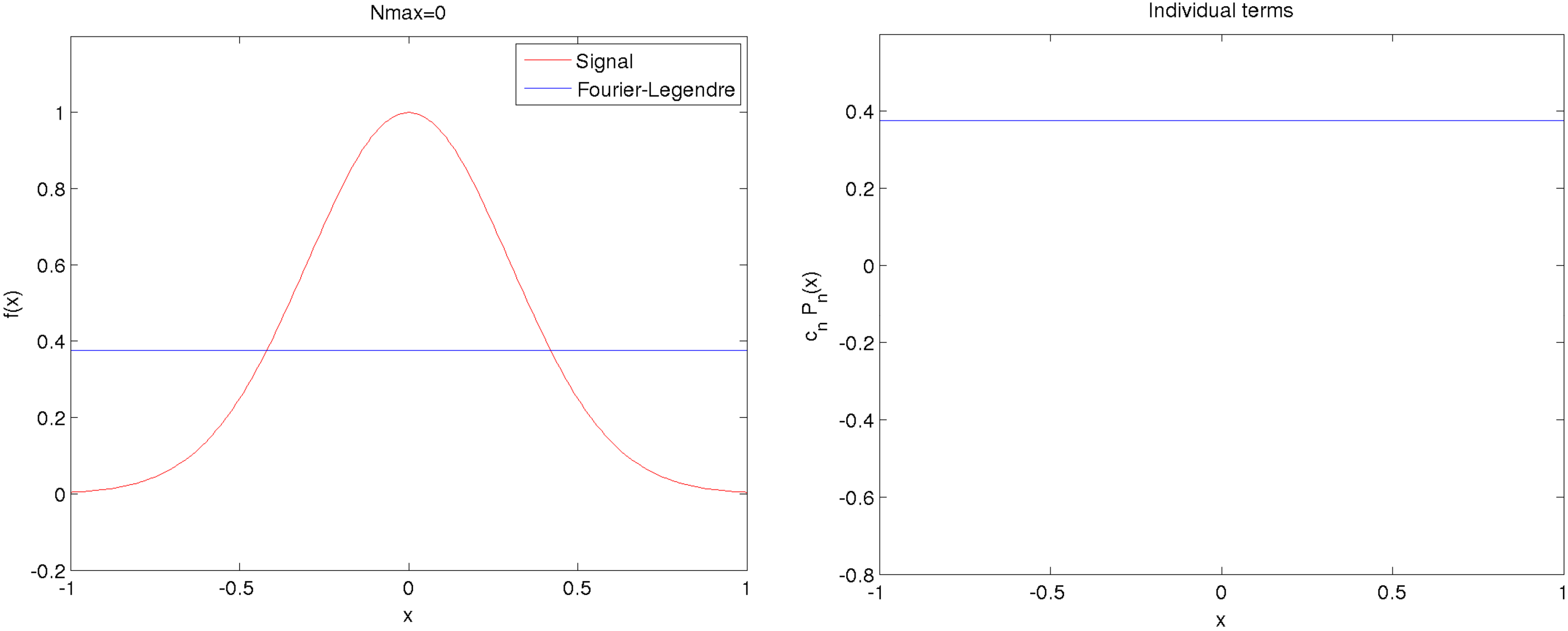
Not orthonormal !

Fourier-Legendre series (for a function *f(x)* square-summable on [-1,1]) :

$$\boxed{f(x)=\sum_{n=0}^{\infty} c_n\,P_n(x)}$$

Coefficient determination :
$$c_n=\left(n+\frac{1}{2}\right)\,(f,P_n)\;=\;\left(n+\frac{1}{2}\right)\,\int_{-1}^{1} f(x)\,P_n(x)dx$$

# Example of Fourier-Legendre reconstruction



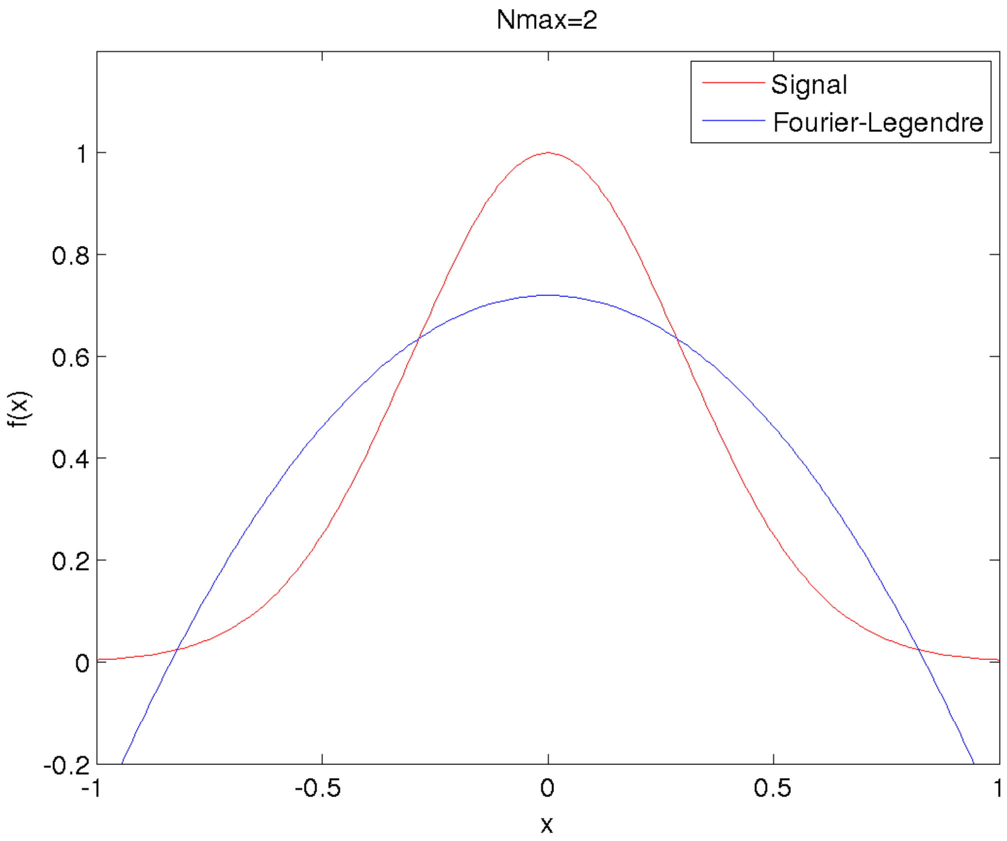
Partial Fourier-Legendre series

Individual term

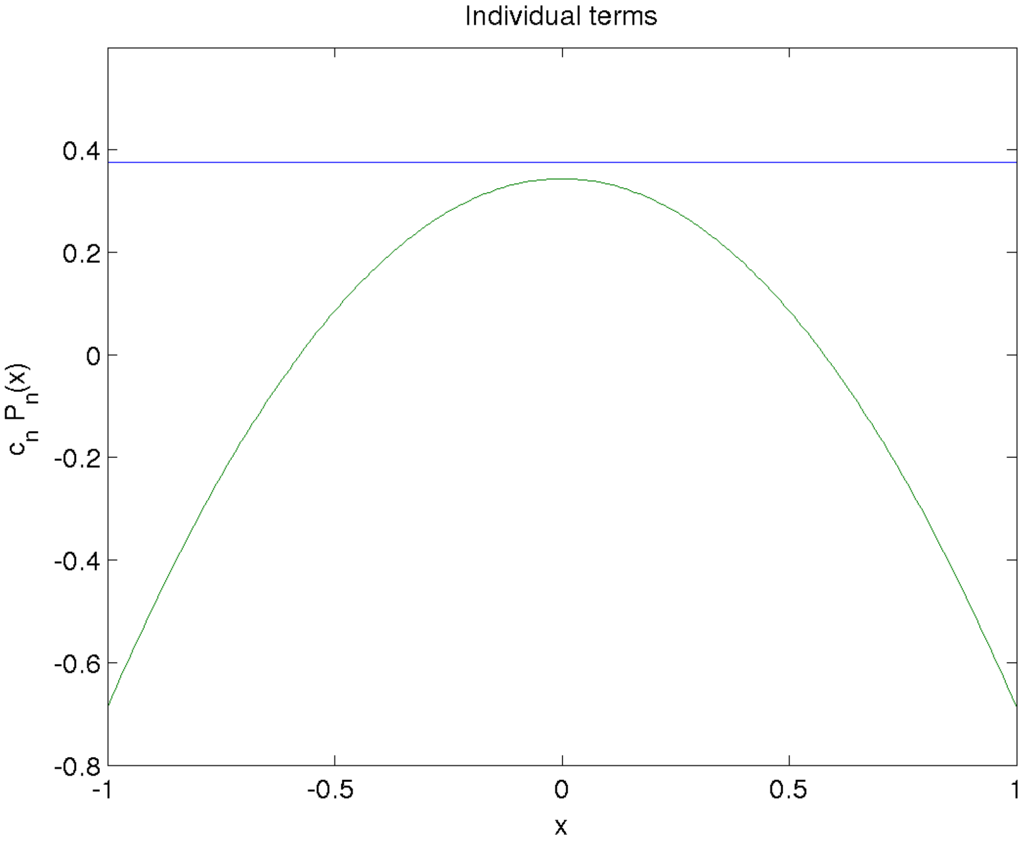
$c_0 P_0(x)$

Signal : $f(x) = \exp\left(-\frac{x^2}{2a^2}\right)$

$a=0.3$



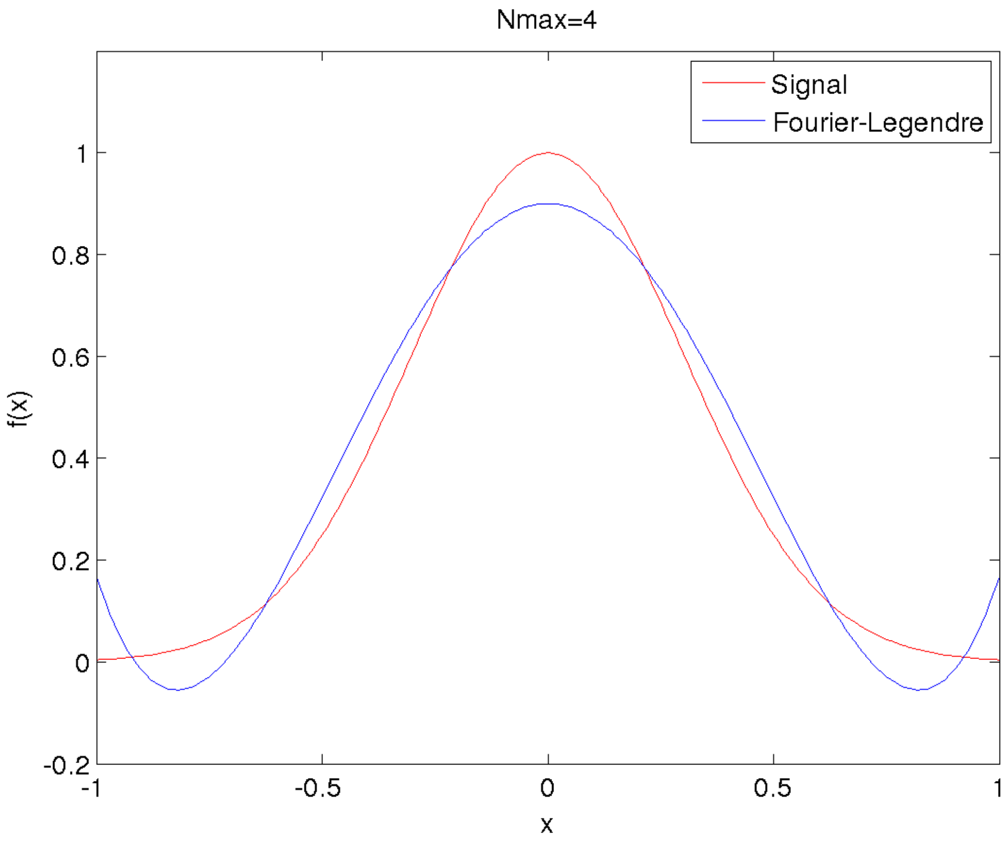
Partial Fourier-Legendre series



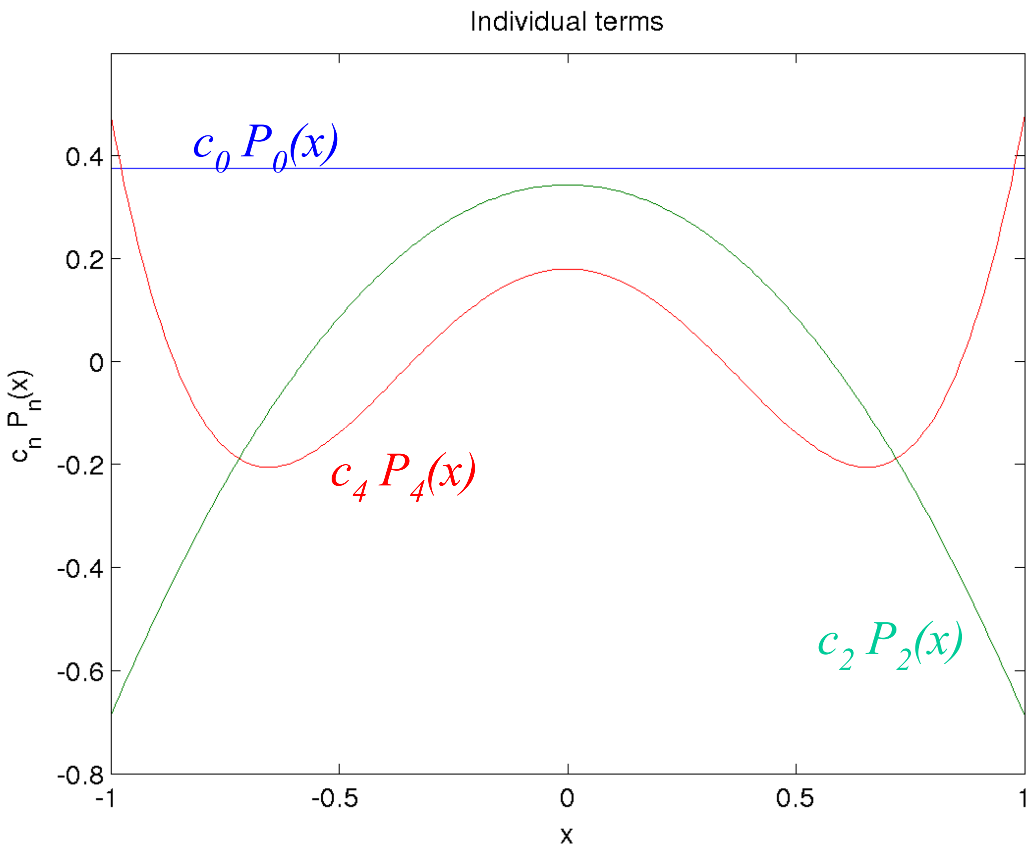
Individual terms

$c_0\, P_0(x)$ and $c_2\, P_2(x)$

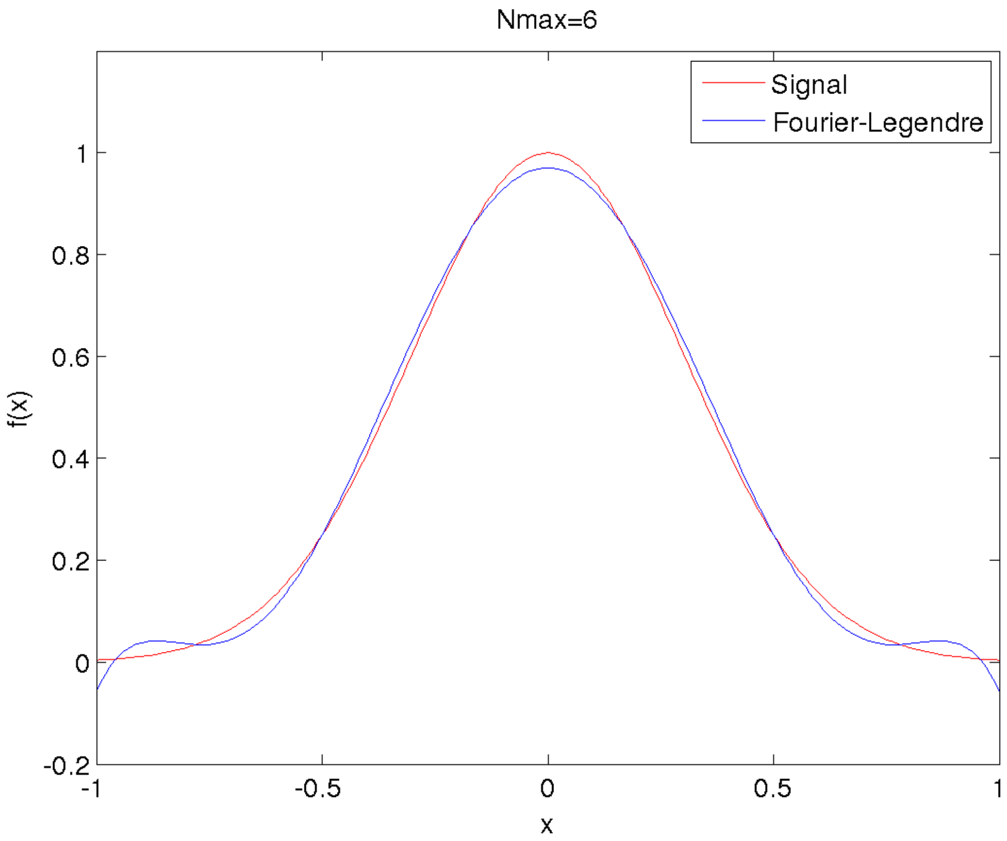
($c_1=0$ for an even signal)



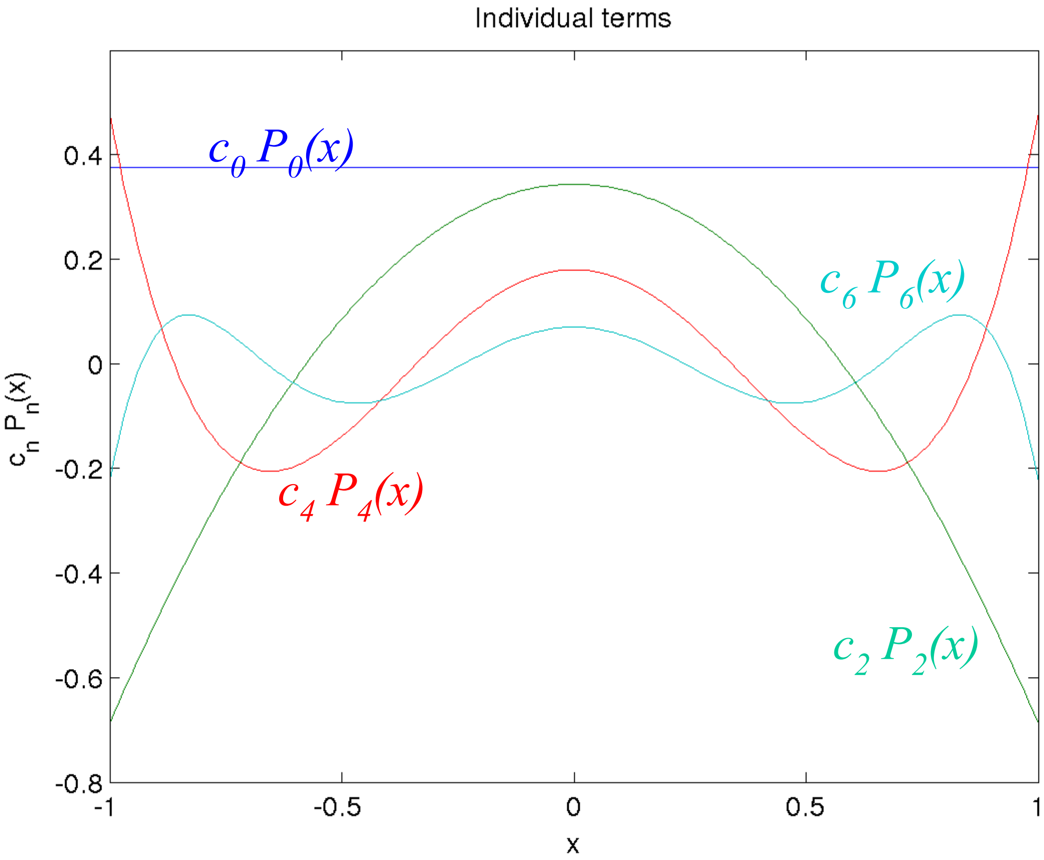
Partial Fourier-Legendre series



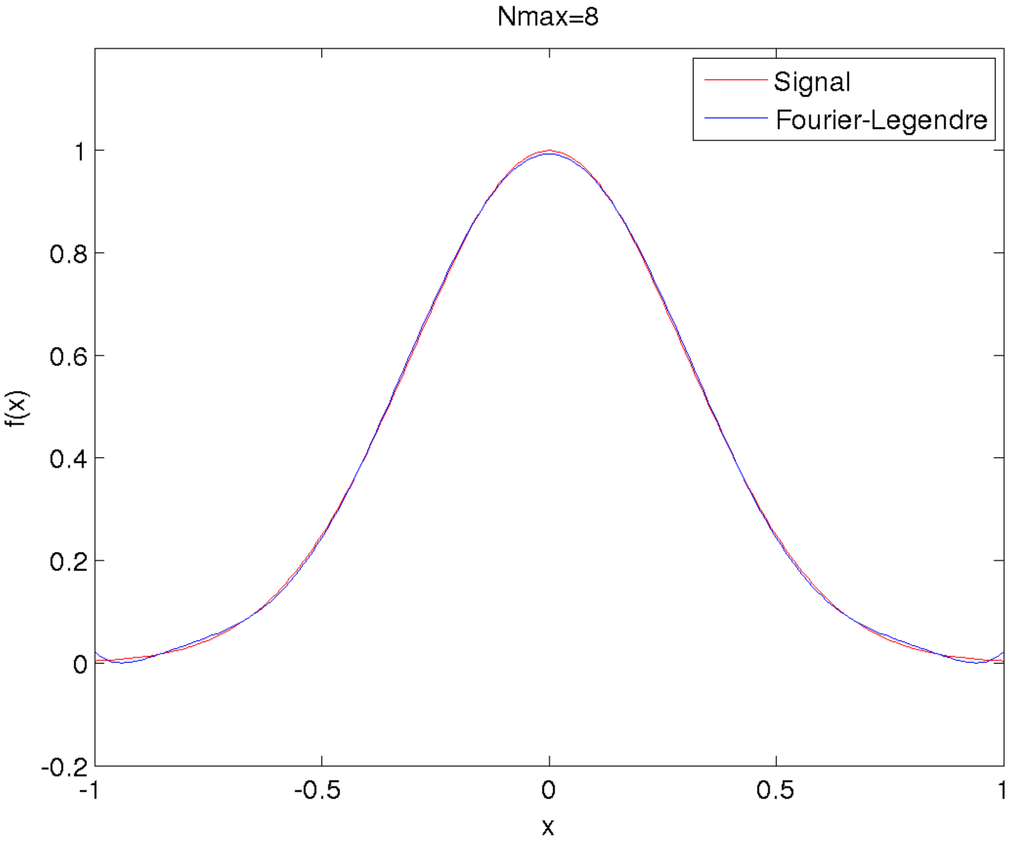
Individual terms



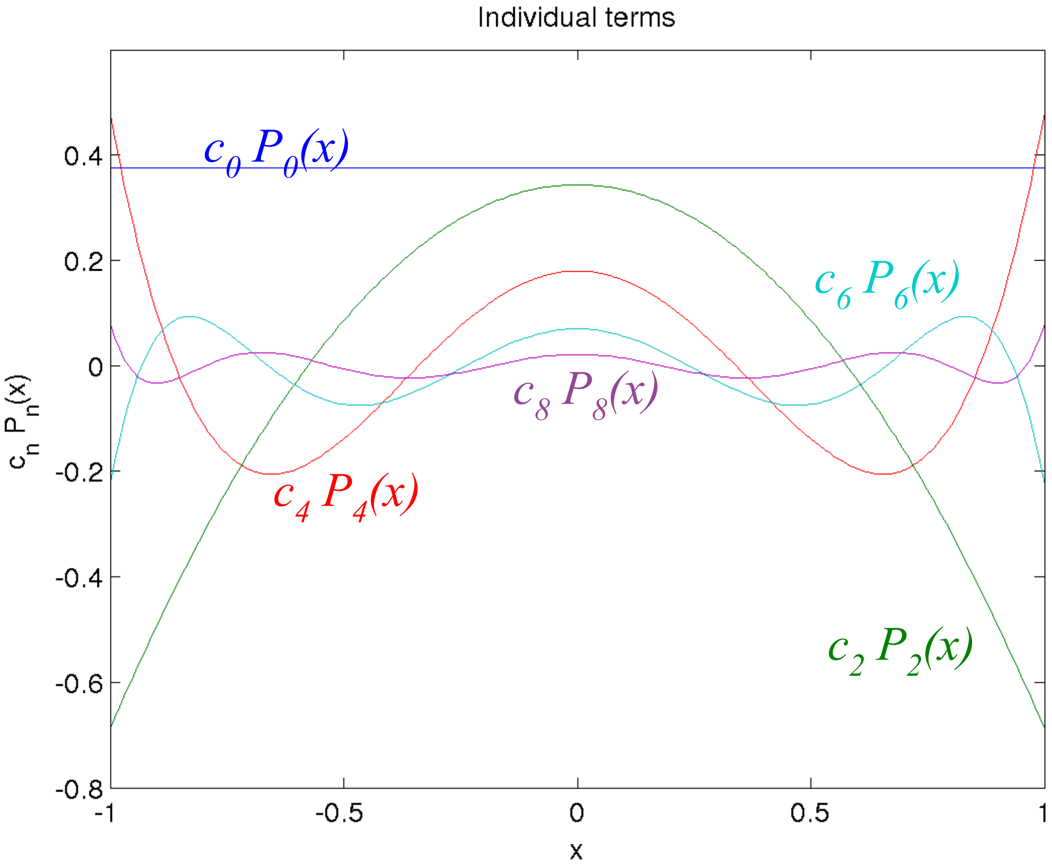
Partial Fourier-Legendre series



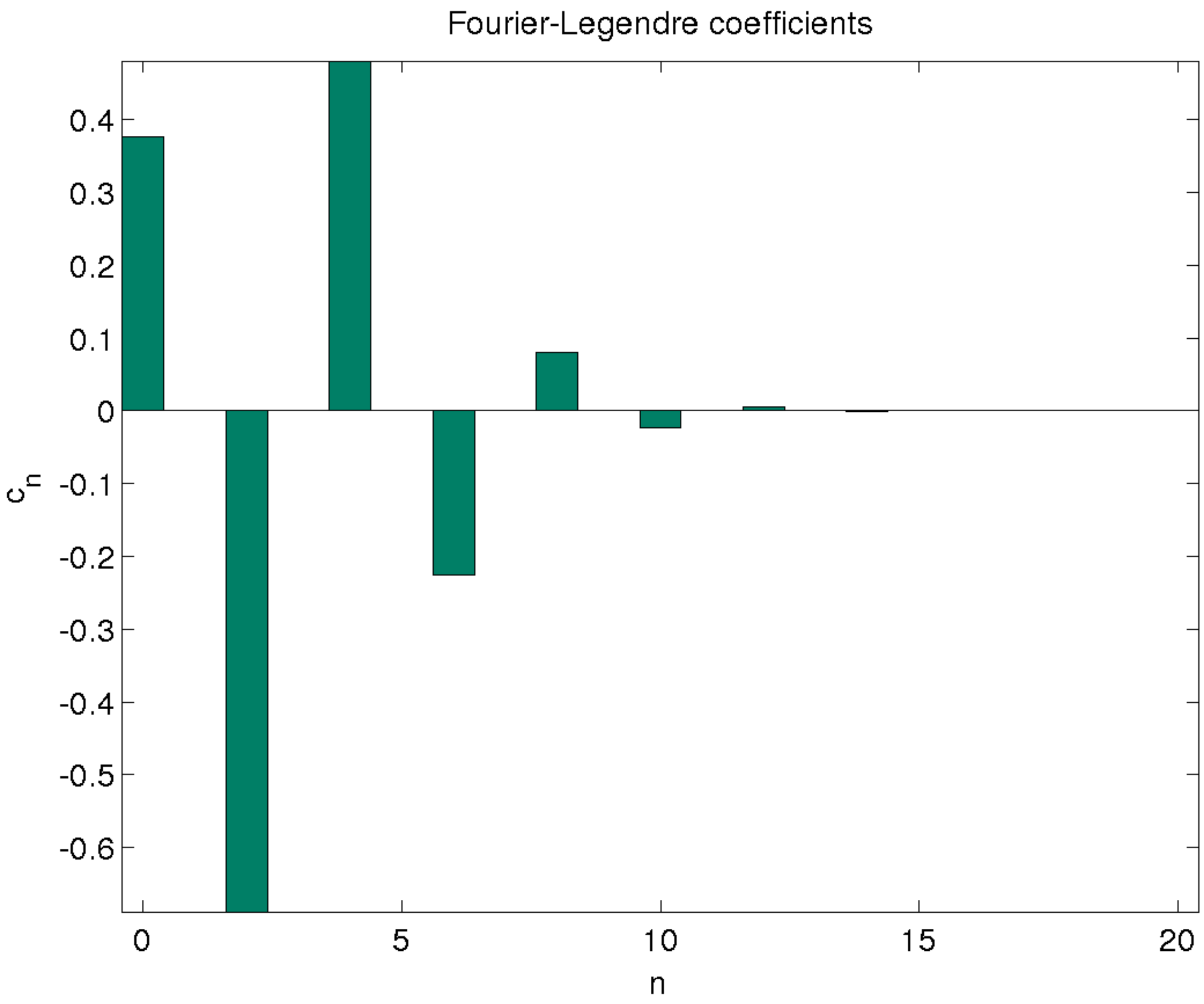
Individual terms



Partial Fourier-Legendre series



Individual terms

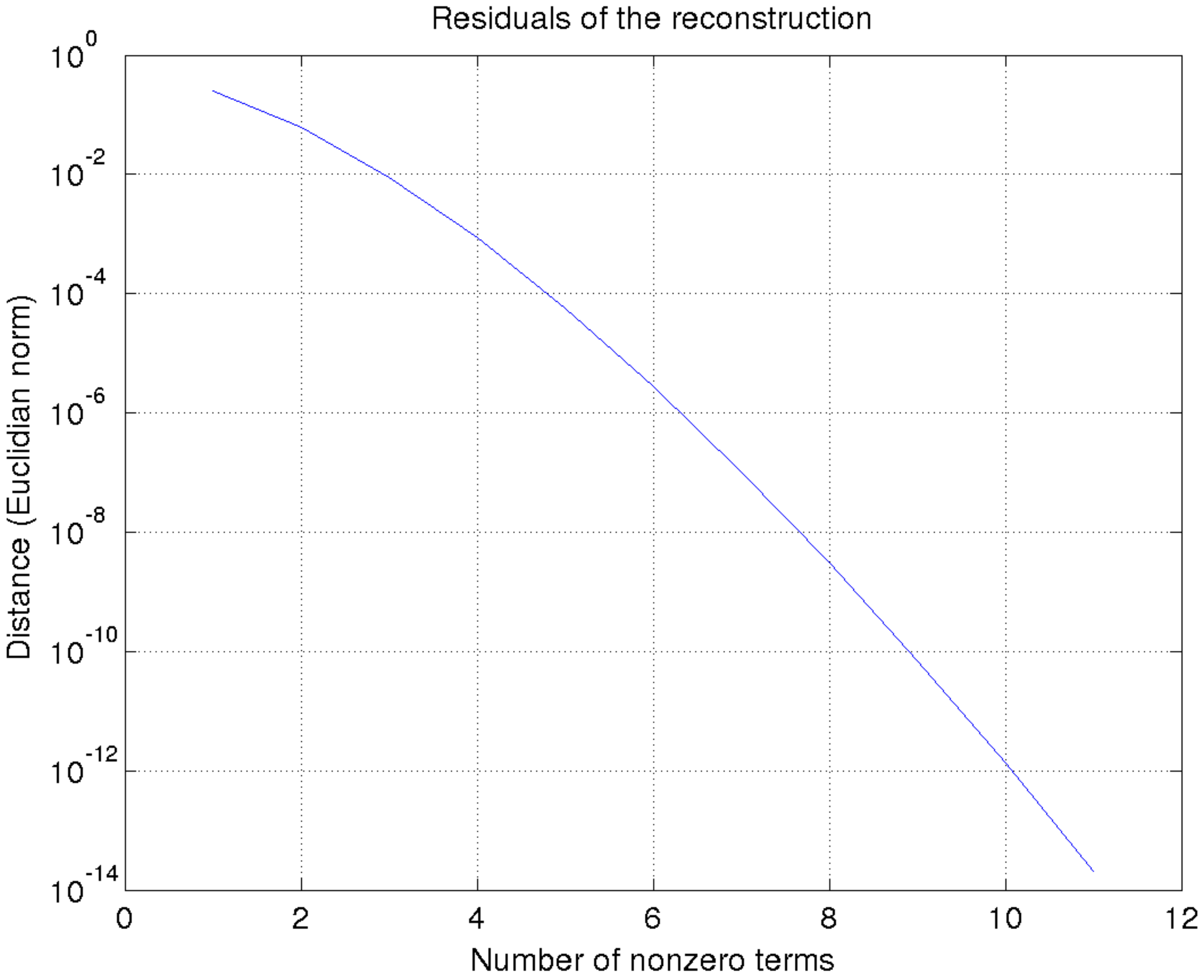


For an even signal : all odd $c_n$ vanish

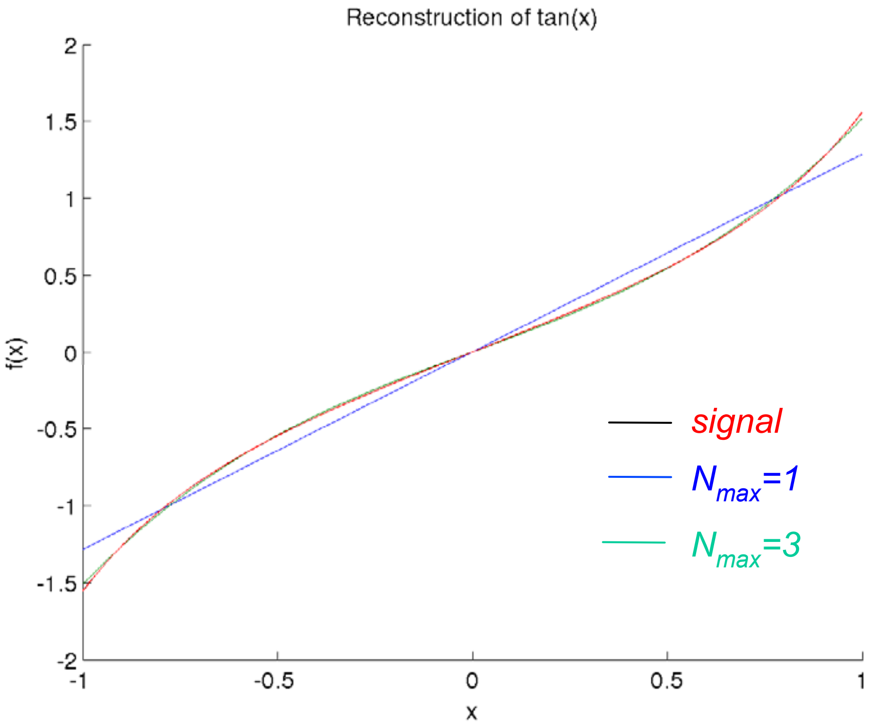
In this example, 8 coefficients seem enough to reconstruct the signal

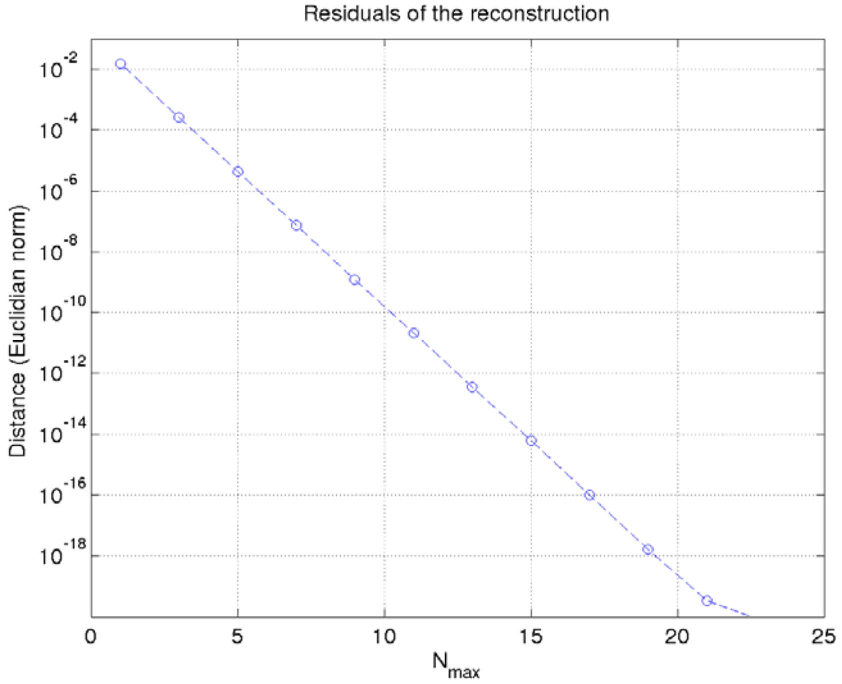
Error on the reconstruction :
Euclidean distance

$$\epsilon = \int_{-1}^{1} \left[ f(x) - \sum_{n=0}^{\boxed{N_{max}}} c_n P_n(x) \right]^2 dx$$

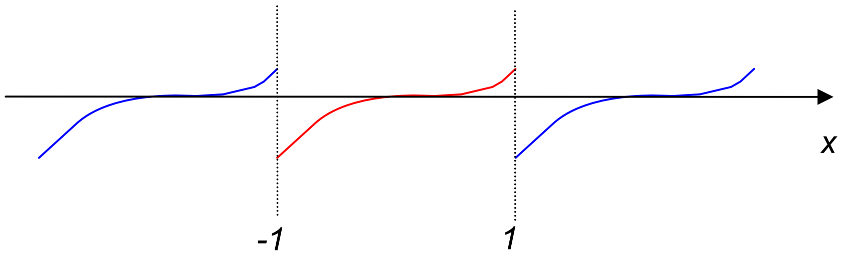


# Another example : $f(x)=tan(x)$

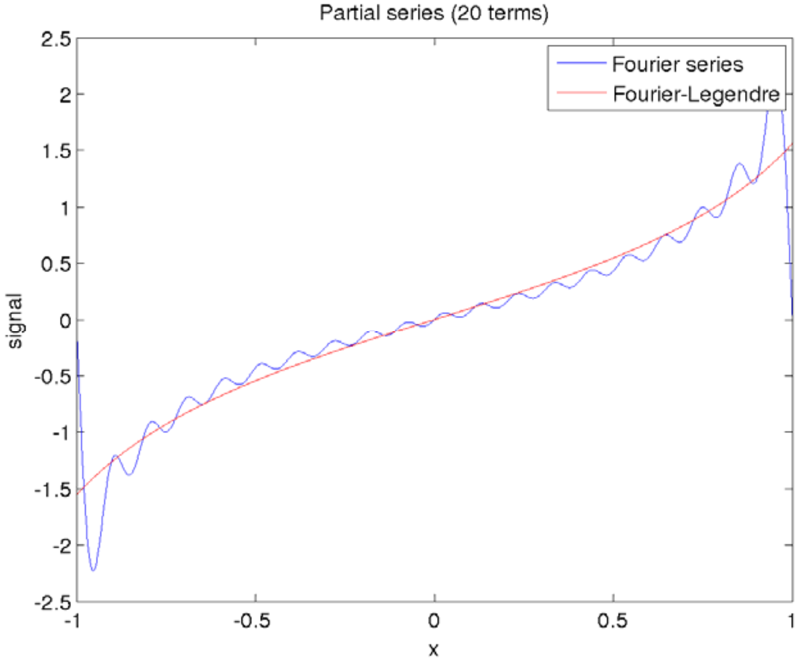




Comparison with Fourier series

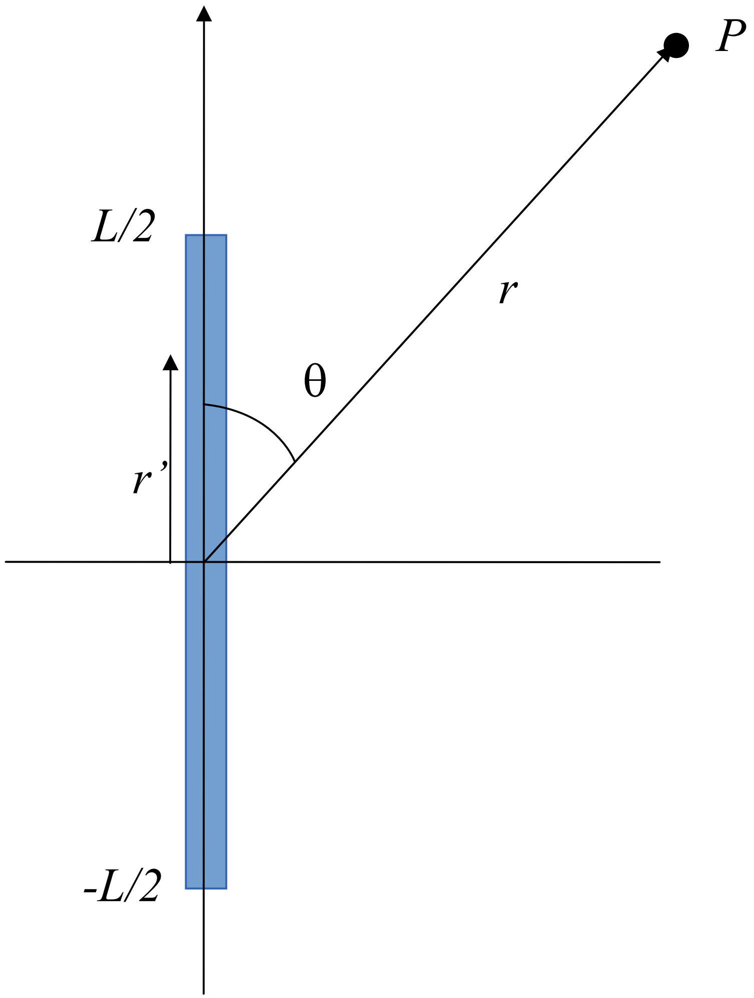


The signal is periodized (period 2)



Advantage to Fourier-Legendre (for this case)

# Example in physics : gravitational potential of a uniform bar



Gravitational potential :

$$\Phi(r,\theta) = G \int_{bar} \frac{\lambda}{|\vec{r}-\vec{r'}|}\, dz$$

λ=linear mass density

Using the generating function of Legendre polynomials:

$$|\vec{r}-\vec{r'}|^{-1} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{z}{r}\right)^n P_n(\cos\theta)$$

One finds easily :

$$\Phi(r,\theta) = G\lambda \sum_{n=0}^{\infty} C_{2n}\, P_{2n}(\cos\theta)$$

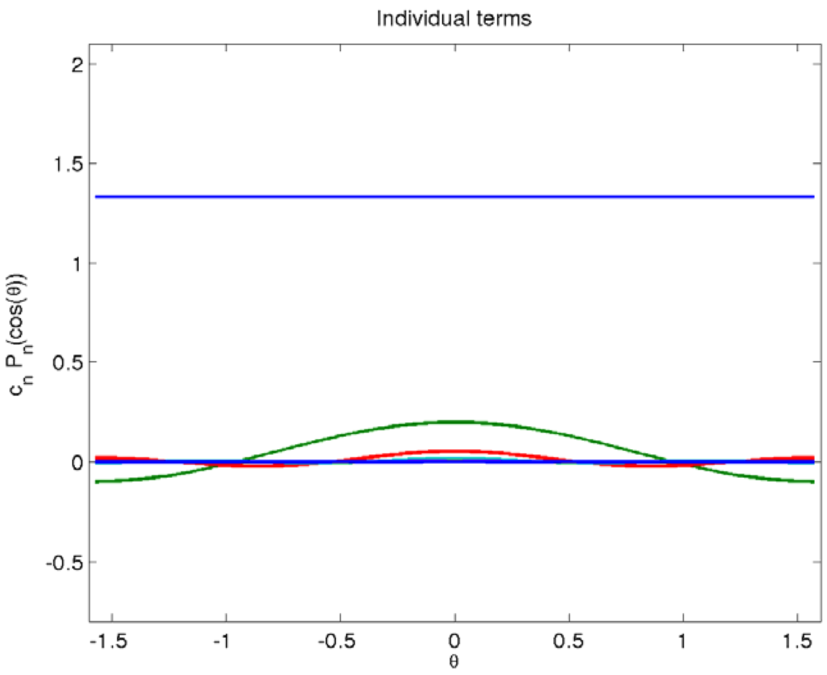
with

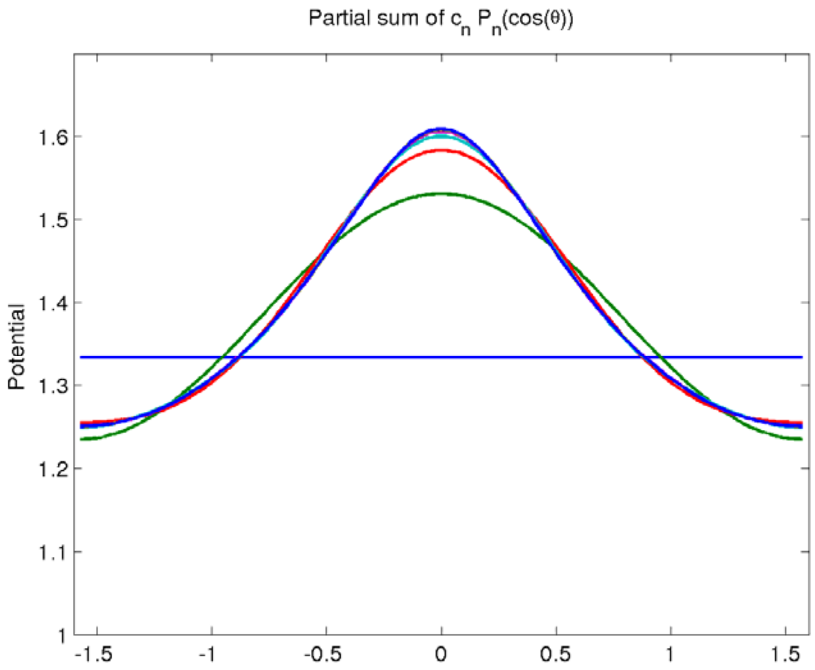
$$C_{2n} = \frac{L^{2n+1}}{(2n+1)\, 2^{2n}\, r^{2n+1}}$$

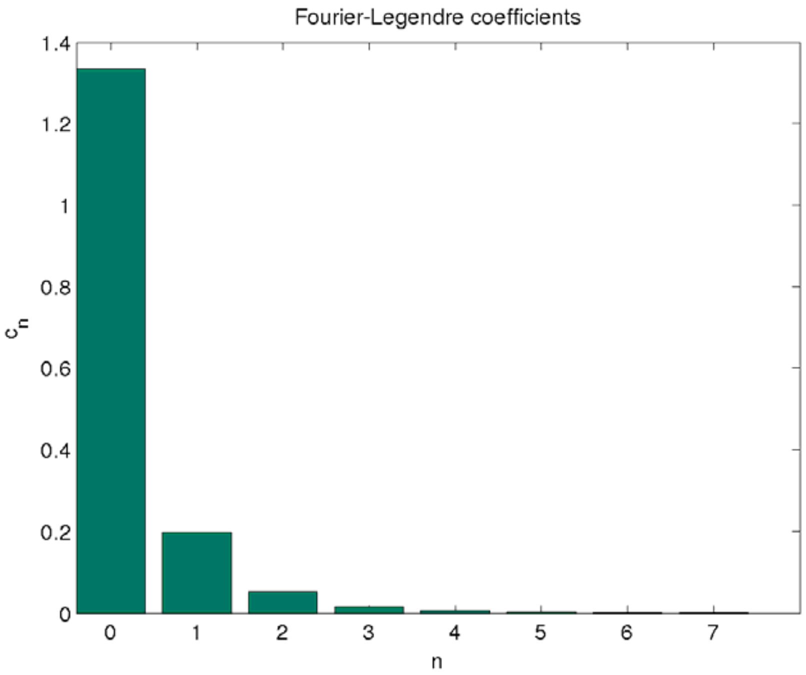
This is a Fourier-Legendre expansion of the potential (a.k.a. multipolar expansion)

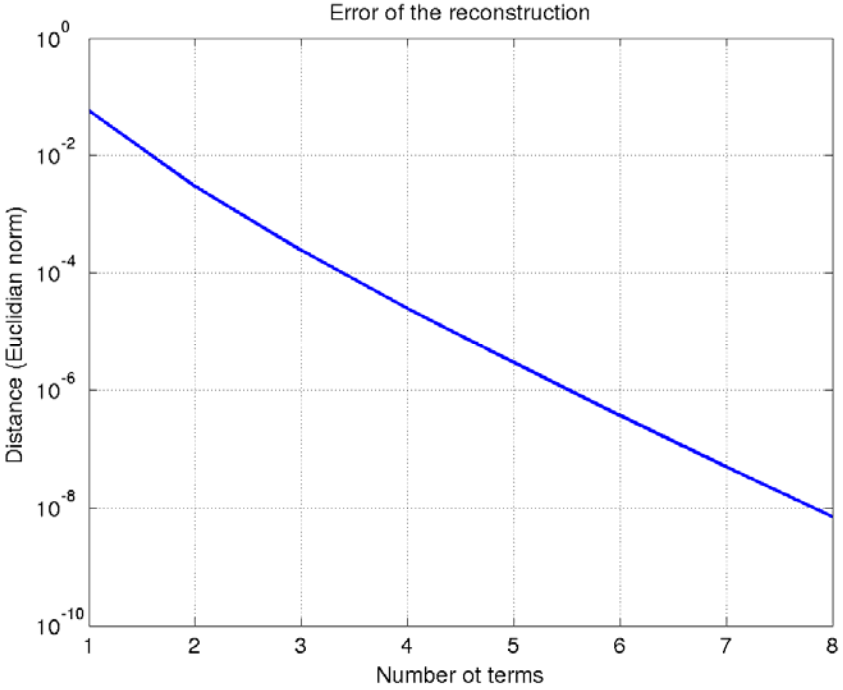
# Fourier-Legendre series of the potential

*r=0.75 L*

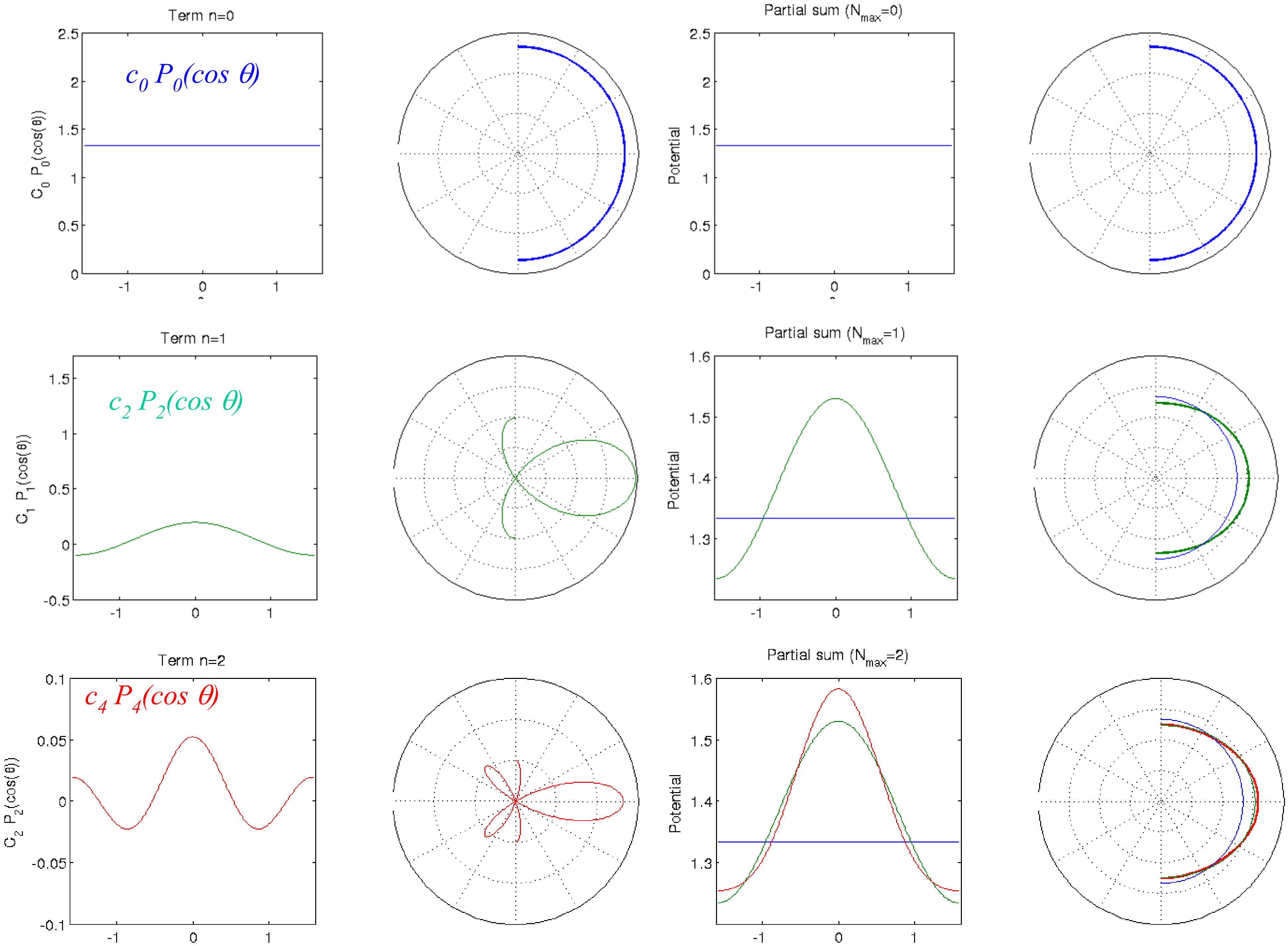








Individual terms Polar plot Potential (partial sums) Polar plot



# Essential ideas on Fourier-Legendre Series

- Signal defined on interval [-1,1]
- Can be expressed as a series of Legendre polynomials (base functions)
- Legendre polynomials are defined by Taylor expansion of a *characteristic function*
- Legendre poly's are orthogonal, with respect to an appropriate scalar product
- Coefs. of the series calculated as a scalar product between the signal and each polynom
- Polynoms are solutions of Legendre differential Eq.

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}P_n(x)\right]+n(n+1)P_n(x)=0.$$

# Representation of signals as series of orthogonal functions

1. Fourier Series
2. Legendre polynomials
3. Spherical harmonics
4. Bessel functions

# Towards the Spherical Harmonics : associated Legendre functions : $P_l^m$

Definition :

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{\mathrm{d}^m}{\mathrm{d}x^m} P_l(x)$$

(for $m>0$)

$-l \leq m \leq l$

$$P_l^{-m} = (-1)^m \frac{(l-m)}{(l+m)!} P_l^m$$

if $m=0$ : $P_l^0 = P_l$ → generalisation of Legendre polynomials

Orthogonality (same scalar product as $P_n$)

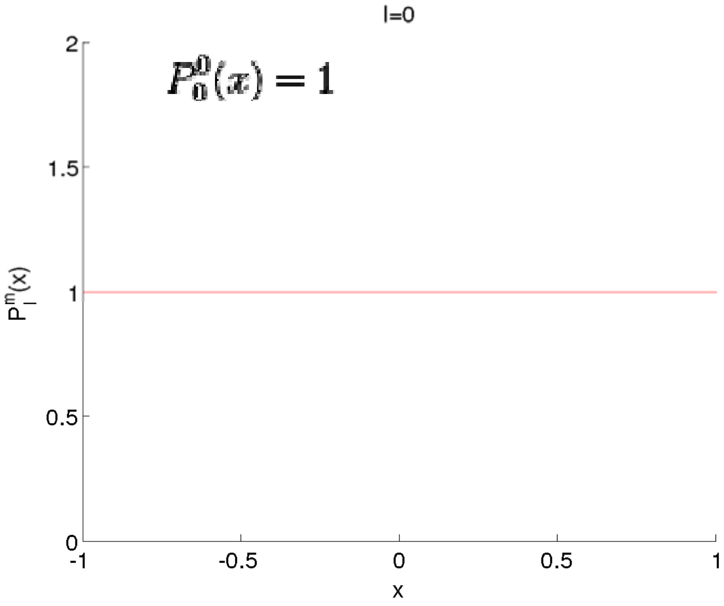
Fixed $m$, different $l$ :

$$(P_l^m, P_{l'}^m) = \frac{2(l+m)!}{(2l+1)!\,(l-m)!}\delta_{ll'}$$

Fixed $l$, different $m$ :

$$(P_l^m, P_l^{m'}) = \frac{(l+m)!}{m\,(l-m)!}\delta_{mm'}$$

## $l=0$ : 1 polynom



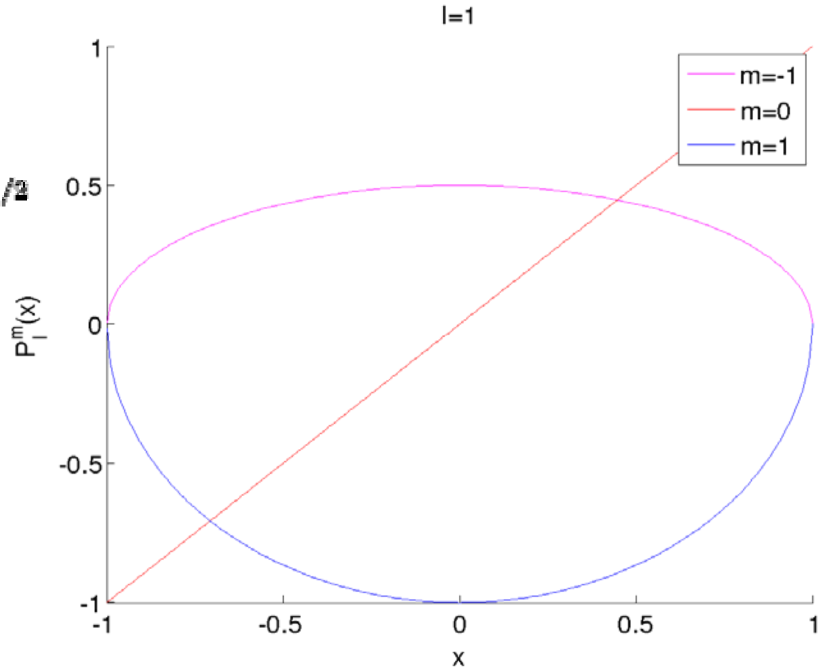
$$P_0^0(x) = 1$$

## $l=1$ : 3 "polynoms"

$$P_1^{-1}(x) = -\frac{1}{2}P_1^1(x)$$

$$P_1^0(x) = x$$

$$P_1^1(x) = -(1-x^2)^{1/2}$$



## $l=2$ : 5 "polynoms"

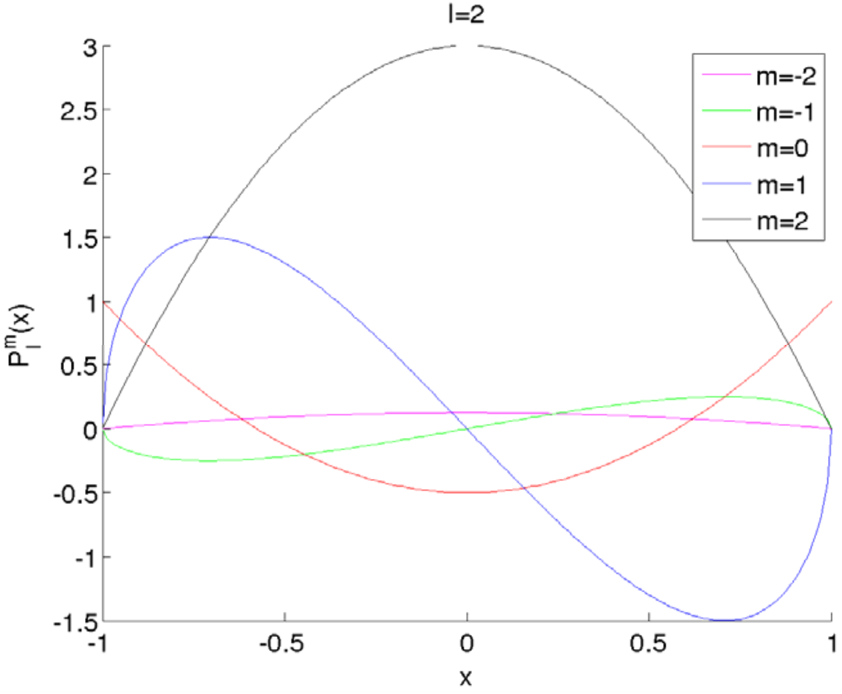
$$P_2^{-2}(x) = \frac{1}{24}P_2^2(x)$$

$$P_2^{-1}(x) = -\frac{1}{6}P_2^1(x)$$

$$P_2^0(x) = \frac{1}{2}(3x^2-1)$$

$$P_2^1(x) = -3x(1-x^2)^{1/2}$$

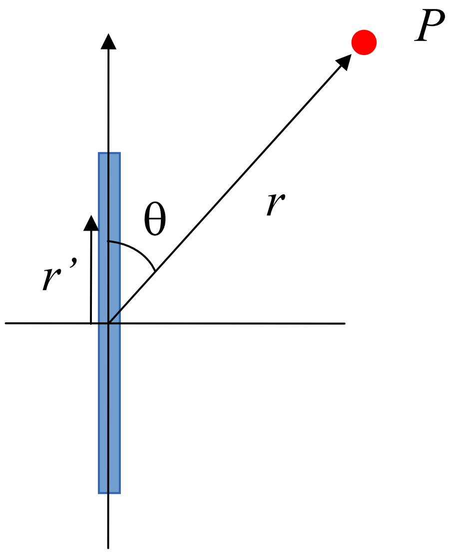
$$P_2^2(x) = 3(1-x^2)$$



*For a given $l$*
*there are $2l+1$*
*possible $P^l_m$*

# Spherical Harmonics

Back to the Newtonial potential :



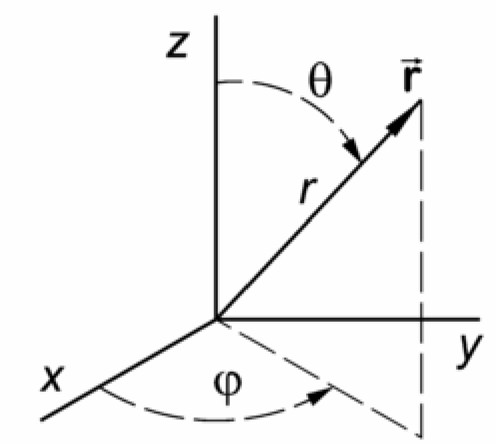
If azimuthal symmetry, the potential is a function of r and θ and can be developped as a Fourier-Legendre series :

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \boxed{c_l(r)}\, P_l(\cos\theta)$$

function of r alone

function of θ alone

What if no azimuthal symmetry ?



The potential depends on the 3 spherical coordinates (r,θ,ϕ). Laplace (1785) showed that a similar series expansion can be made :

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \boxed{C_{lm}(r)}\, \boxed{Y_l^m(\theta,\phi)}$$

# Spherical Harmonics

To find the functions $C_{lm}(r)$ and $Y_l^m(\theta,\phi)$, say that the potential obey Laplace's equation

$$\Delta\Phi = 0$$

(Solutions are « harmonic functions »)

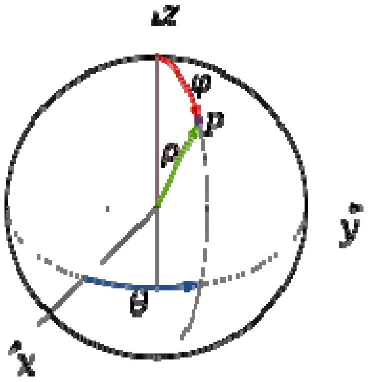
and find :

$$\boxed{Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\phi}}$$

Spherical Harmonics

$l$ : degree, $m$ : order ; $-l \leq m \leq l$

The potential at the surface of a sphere (fixed r) is a 2D function $f(\theta, \phi)$ which can be developped as a series of spherical harmonics :



$$\Phi(r,\theta,\phi) = \boxed{f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\, Y_l^m(\theta,\phi)}$$

# Orthogonality and series expansion

Scalar product :

(suited to Spherical harmonics)

$$(f,g) = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(\theta,\phi)\, \overline{g(\theta,\phi)}\, \sin\theta \, d\theta \, d\phi$$

Orthogonality :

$$(Y_l^m, Y_{l'}^{m'}) = \boxed{\delta_{mm'}\delta_{ll'}}$$

*orthonormal base*

Series expansion for a function *f(θ,ϕ)* on a sphere :

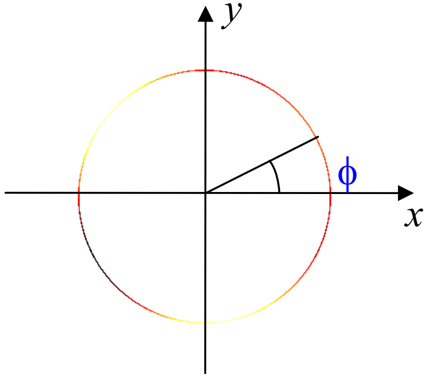
$$f(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, Y_l^m(\theta,\phi)$$

Coefficient determination :

$$a_{lm} = (f, Y_l^m) = \int_0^{\pi} \int_0^{2\pi} f(\theta,\phi)\, \overline{Y_l^m(\theta,\phi)}\, \sin\theta \, d\theta \, d\phi$$

# Comparison : 1D (polar) and 2D (spherical)

In the plane : polar coordinates *(r,ϕ)*



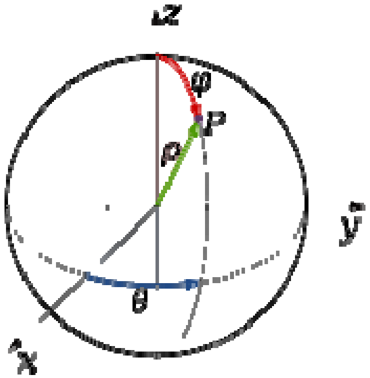
Function *f(ϕ)* with values on a circle $r=C^{te}$

2π periodic in ϕ

Fourier expansion

$$f(\phi) = \sum_{n=-\infty}^{\infty} c_n e^{in\phi}$$

In space : spherical coordinates *(r,θ,ϕ)*



Function *f(θ,ϕ)* with values on a sphere $r=C^{te}$

2π periodic in ϕ and θ

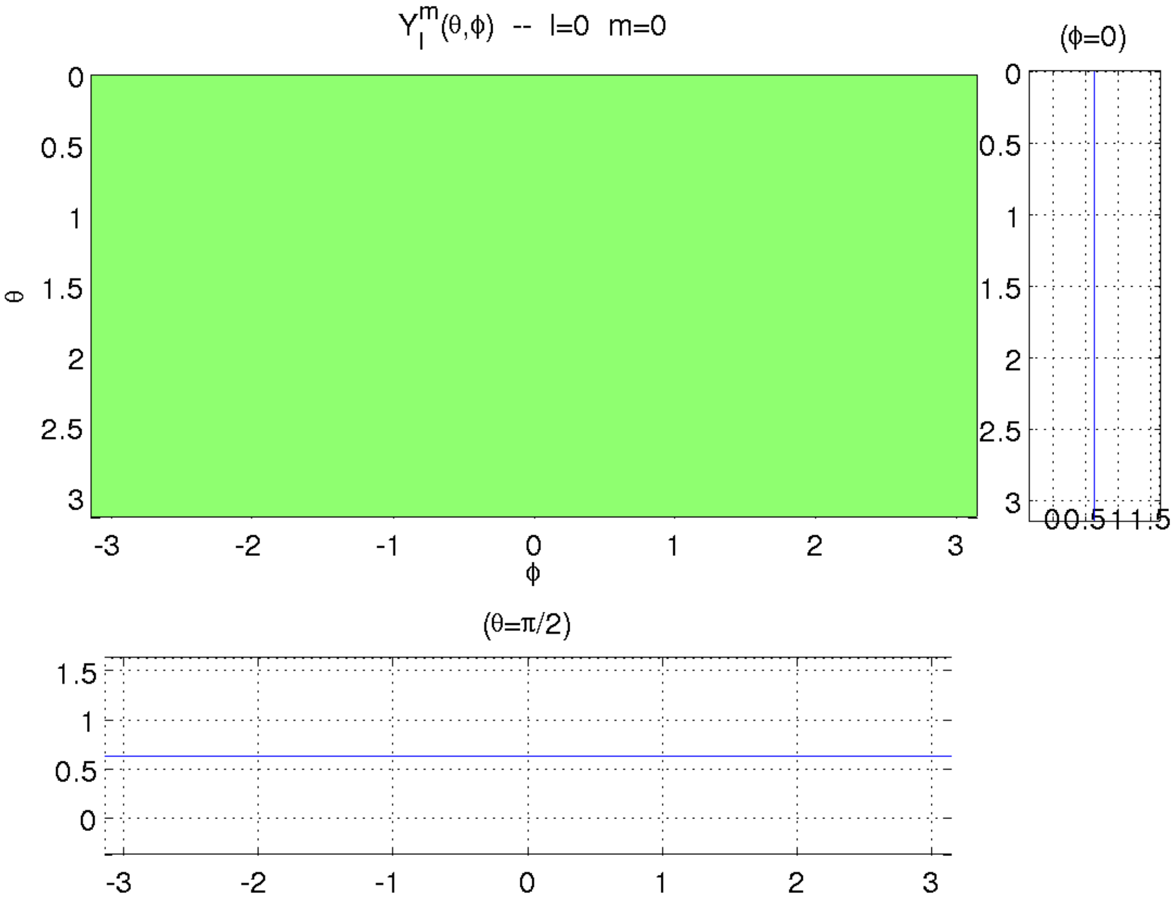
Spherical harmonic expansion

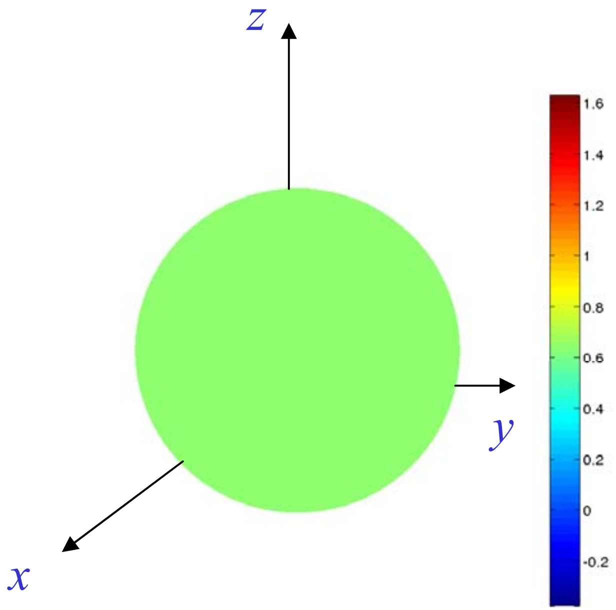
$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_l^m(\theta,\phi)$$

# First spherical harmonic

$$Y_0^0(\theta,\varphi)=\frac{1}{2}\sqrt{\frac{1}{\pi}}$$

(uniform)

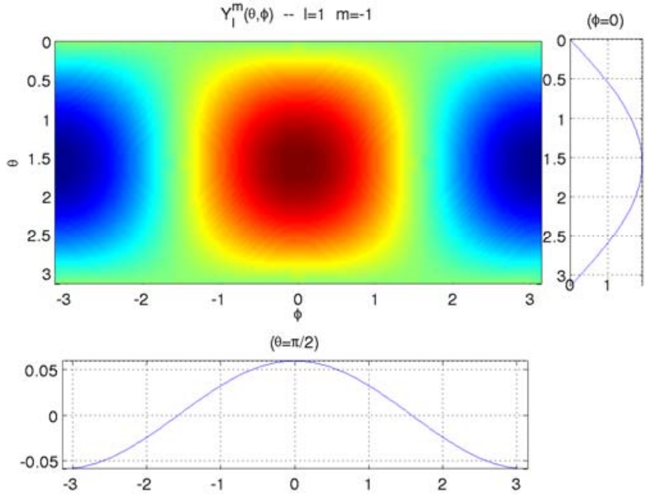


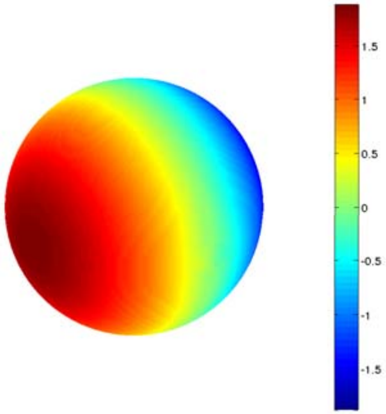


# l=1

## m=-1

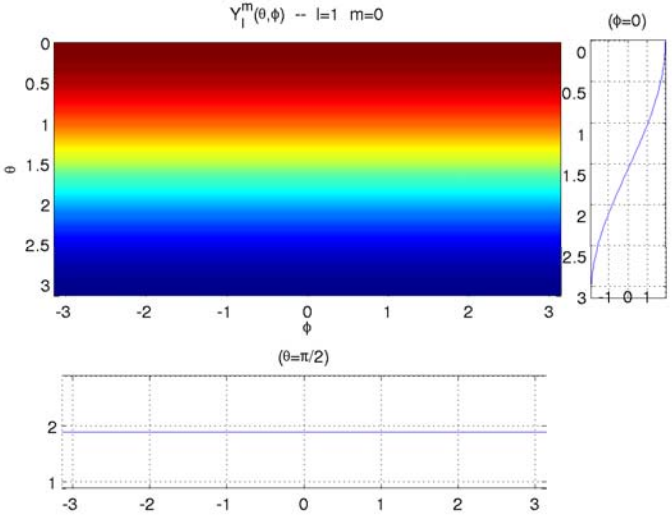
$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\, e^{-i\varphi}$$

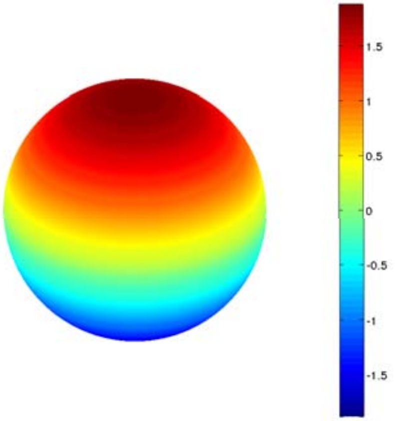




## m=0

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}}\,\cos\theta$$

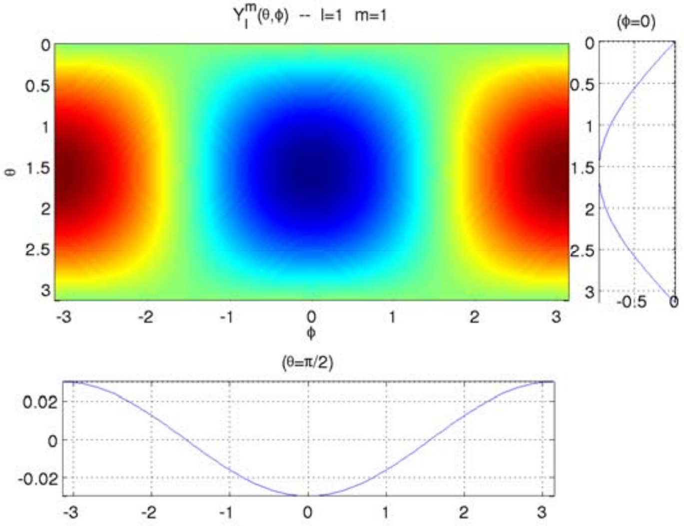


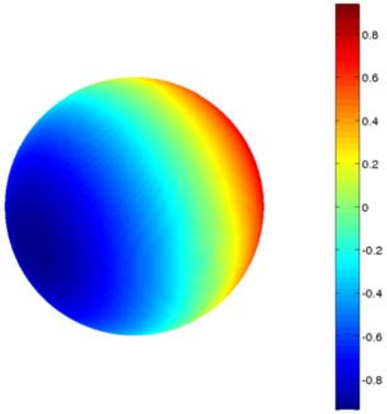


indep. of ϕ : azimuthal symmetry

## m=1

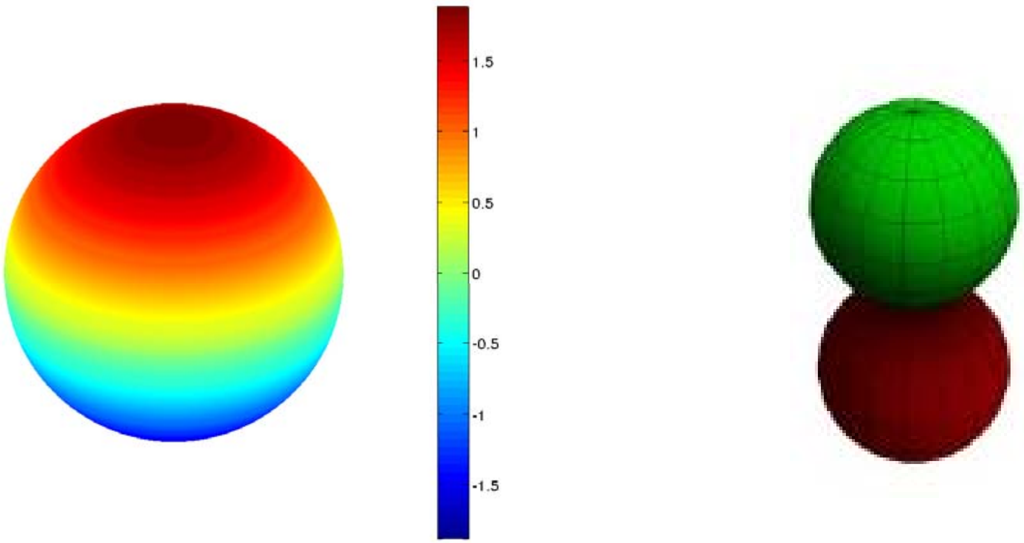
$$Y_1^{1}(\theta,\varphi) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\,\sin\theta\, e^{i\varphi}$$



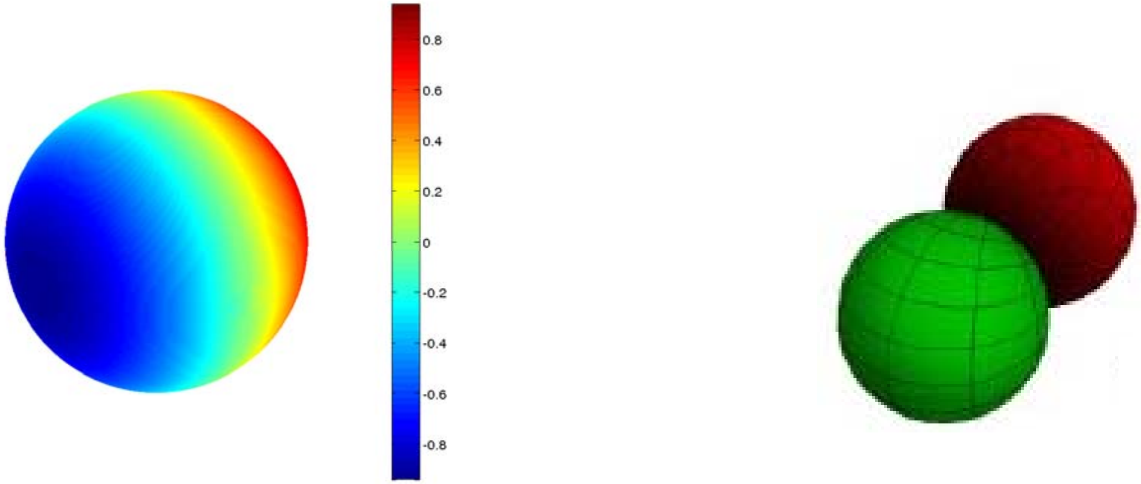


# Different representation

l=1, m=0



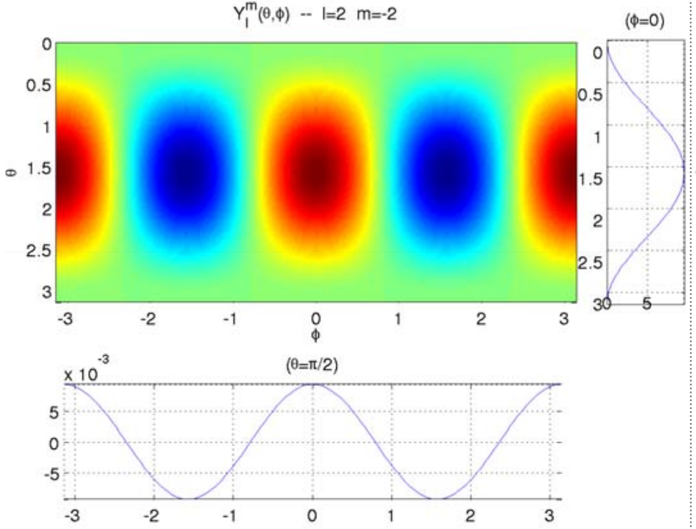
l=1, m=1

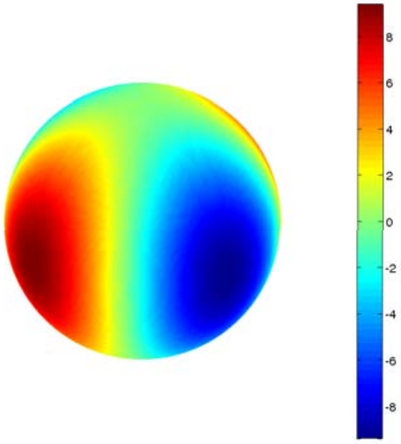


# l=2

## m=-2

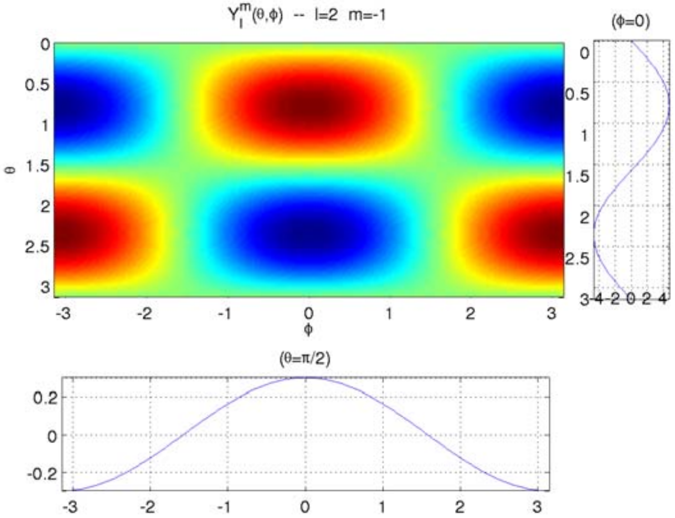
$$Y_2{}^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{\;\;2i\varphi}$$

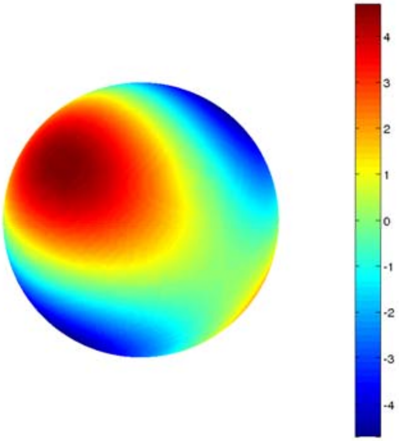




## m=-1

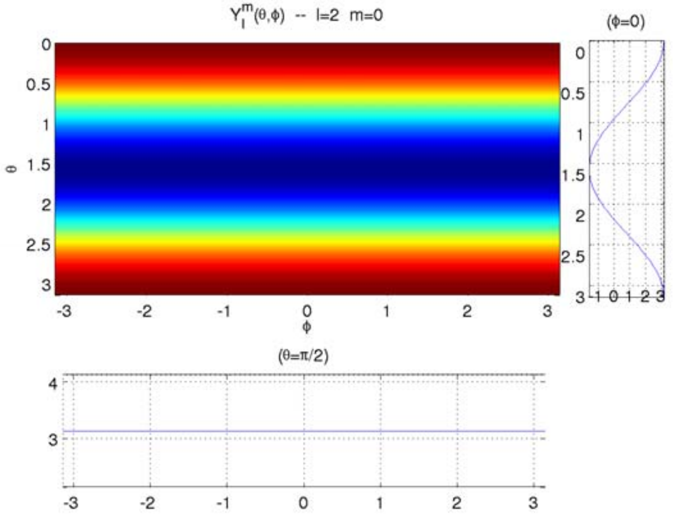
$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}\,\sin\theta\,\cos\theta\, e^{-i\varphi}$$

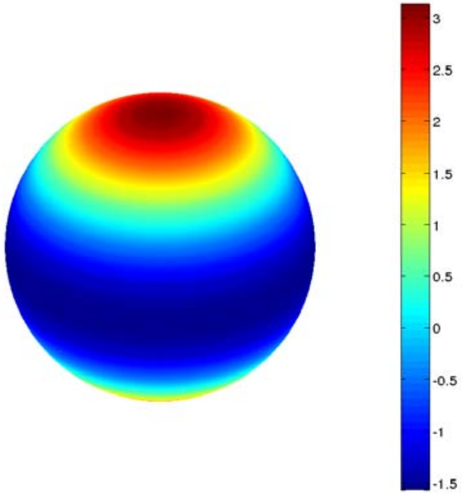




## m=0

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\,(3\cos^2\theta - 1)$$





for positive m, use : $Y_l^{-m}(\theta,\phi) = (-1)^m\,\overline{Y_l^m(\theta,\phi)}$

# Symmetries of the spherical harmonics

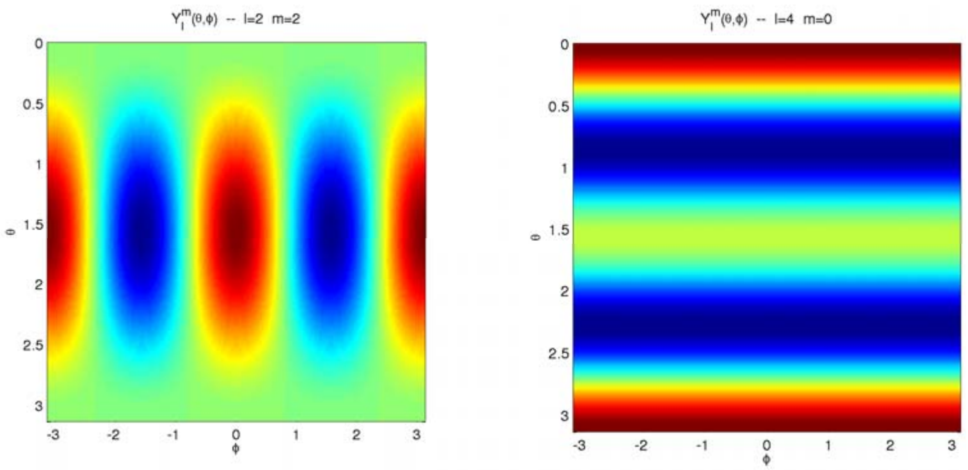
*m=0* : azimuthal symmetry

$$Y_l^0(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\, P_l(\cos\theta)$$

Spherical harmonic expansion becomes a Fourier-Legendre series of $\cos(\theta)$

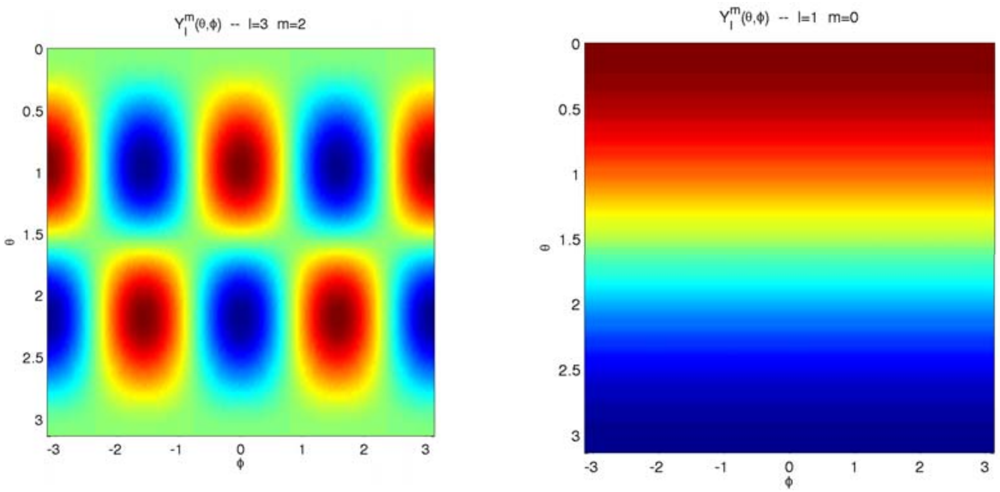
Parity in $\cos(\theta)$ :

*l+m* even : symmetry plane $z=0$



Adapted for series expansion of functions having a symmetry plane

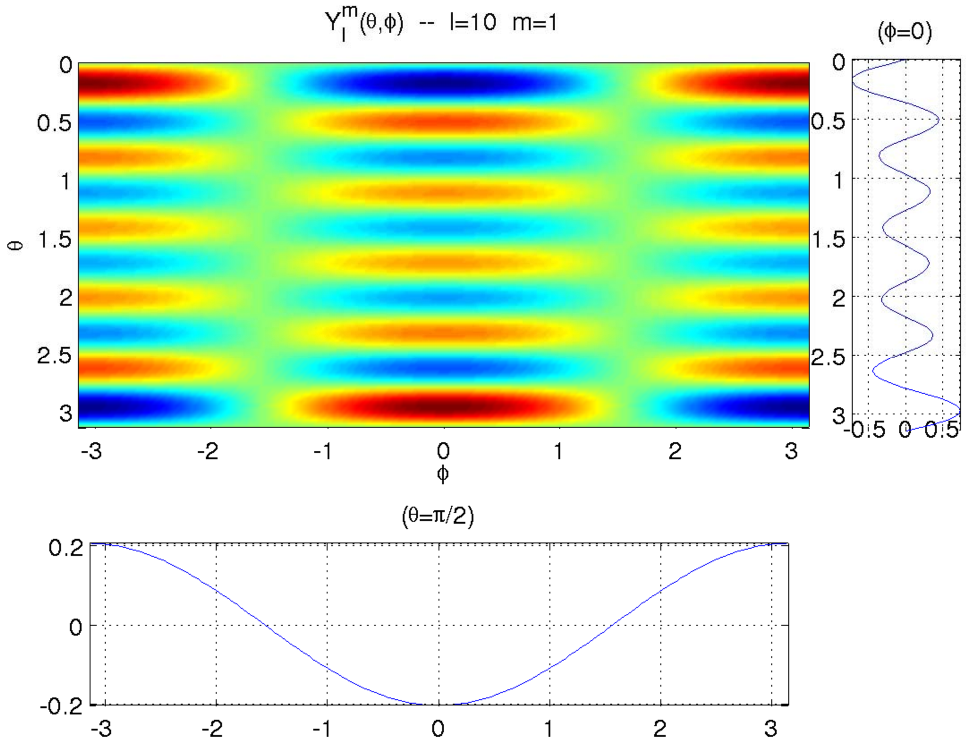
*l+m* odd: anti-symmetry plane $z=0$

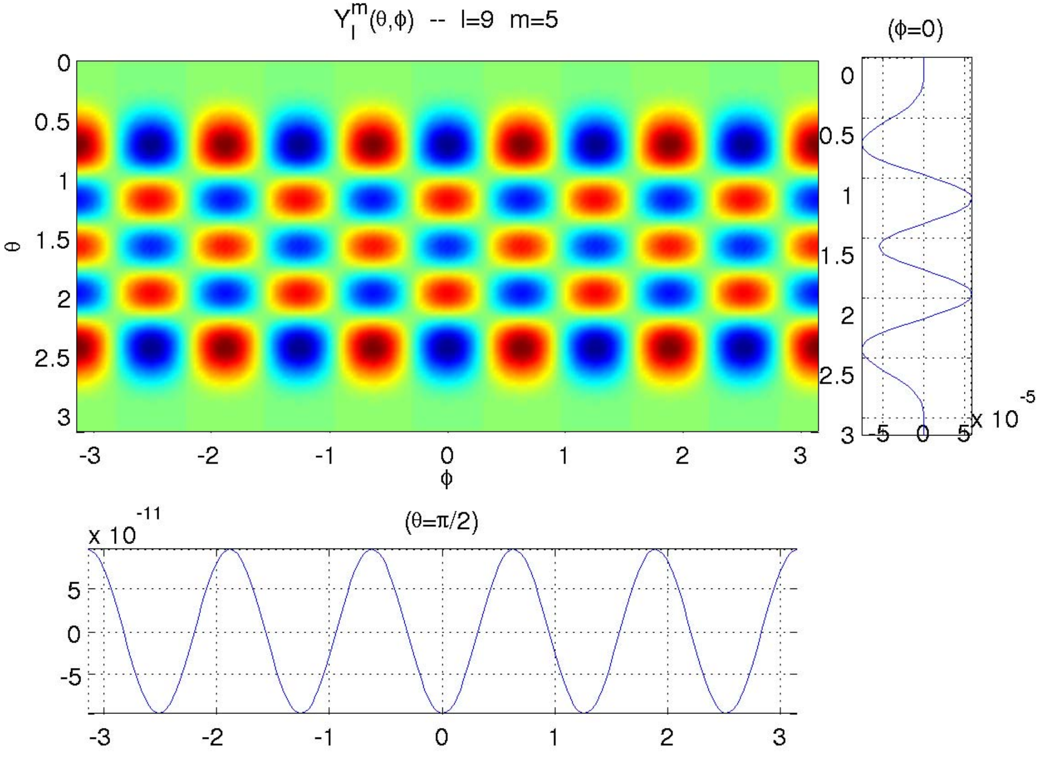


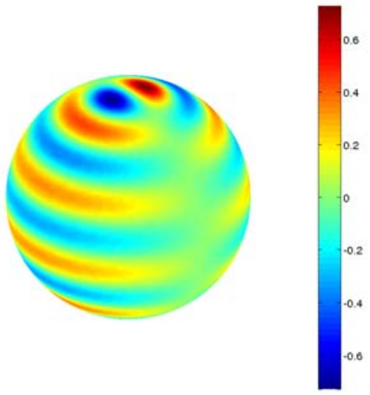
Adapted for series expansion of functions having a anti-symmetry plane

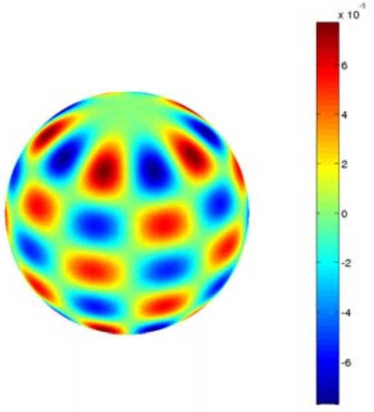
# Harmonic $Y_l^m$ : $m$ periods in direction $\phi$ (term $e^{im\phi}$)
# $n$ - $m$ node lines ($Y_l^m=0$) in direction $\theta$

Large $m$ or $(l\text{-}m)$
= small details









# Application to wide-field imagery : WMAP images

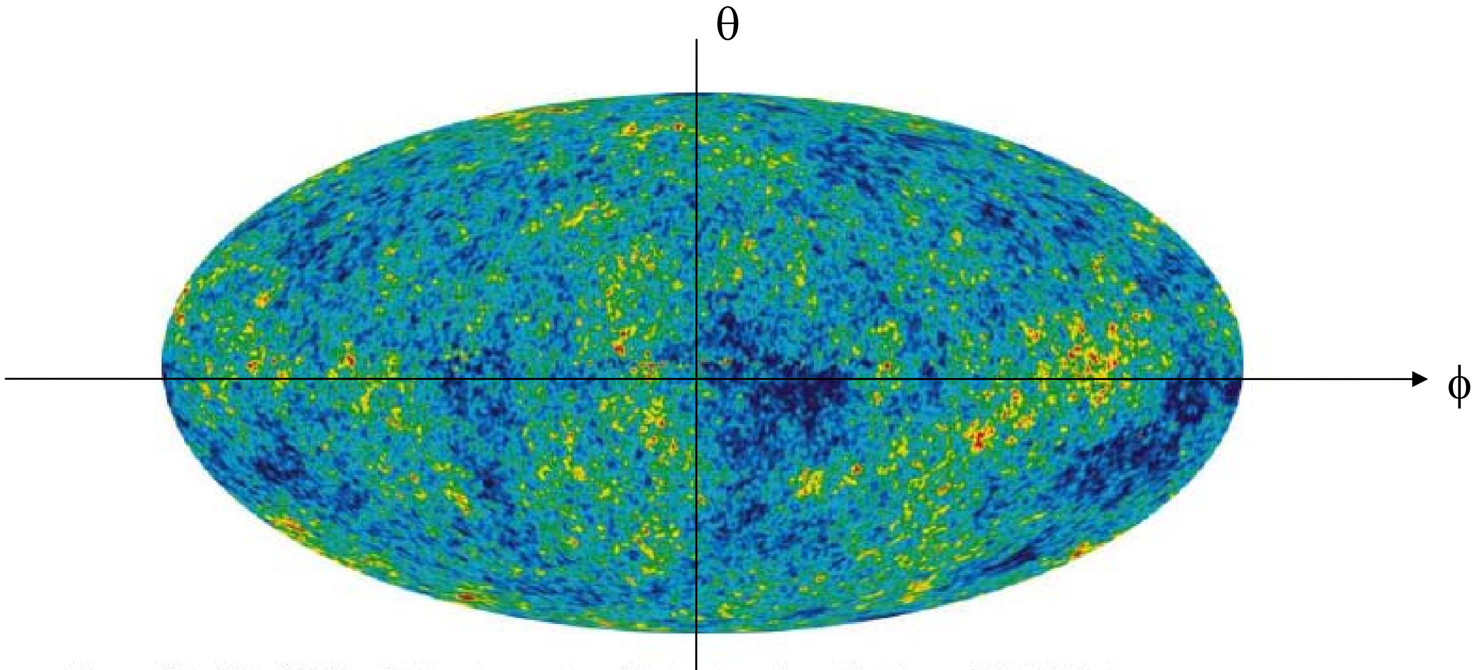
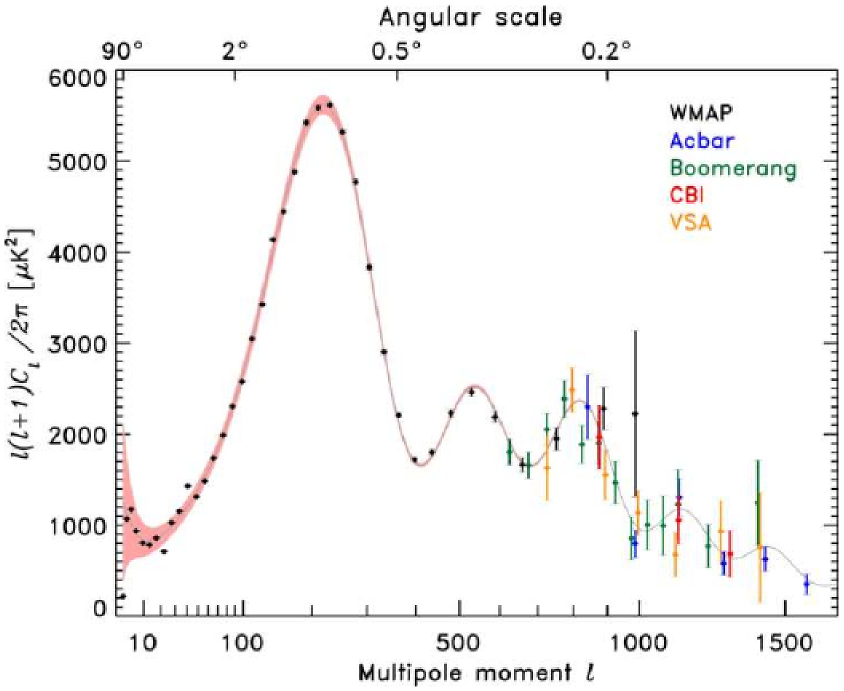


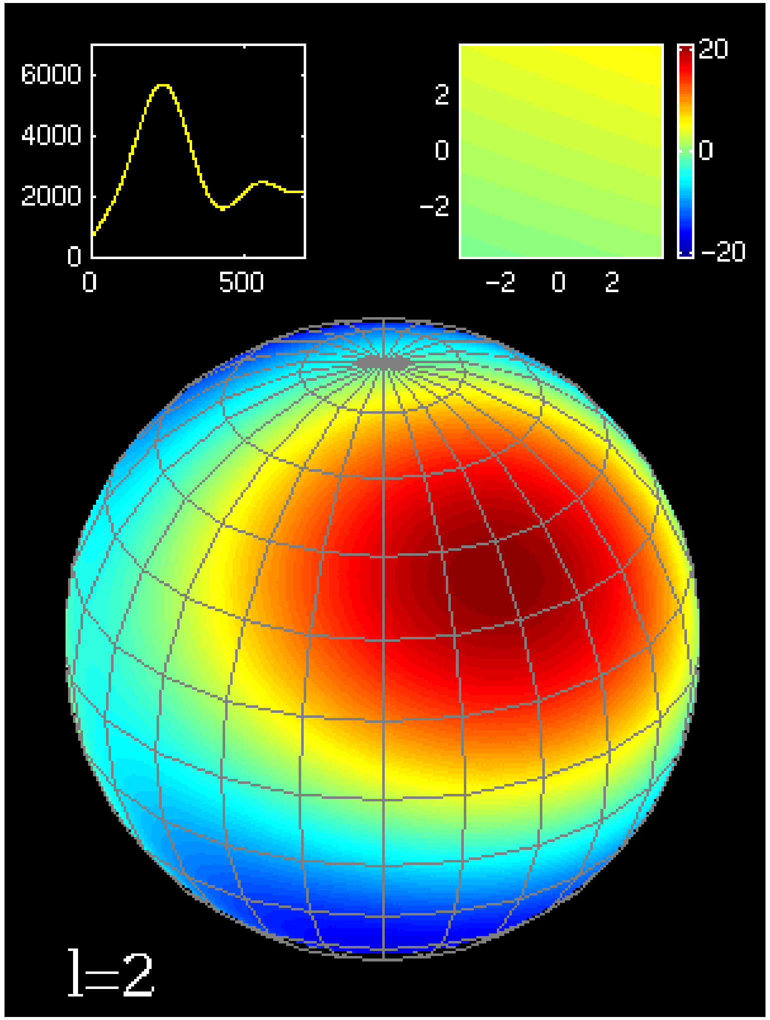
Figure 32: The CMB radiation temperature fluctuations from the 5-year WMAP data

Decomposition of the CMB signal on the $Y^l_m$
Plot of coefficients $|c_l|^2$ (averaged over $m$),
i.e. « angular power spectrum »



*Source : B. Terzic,*
*http://www.nicadd.niu.edu/~bterzic/*

# Application to wide-field imagery : WMAP images

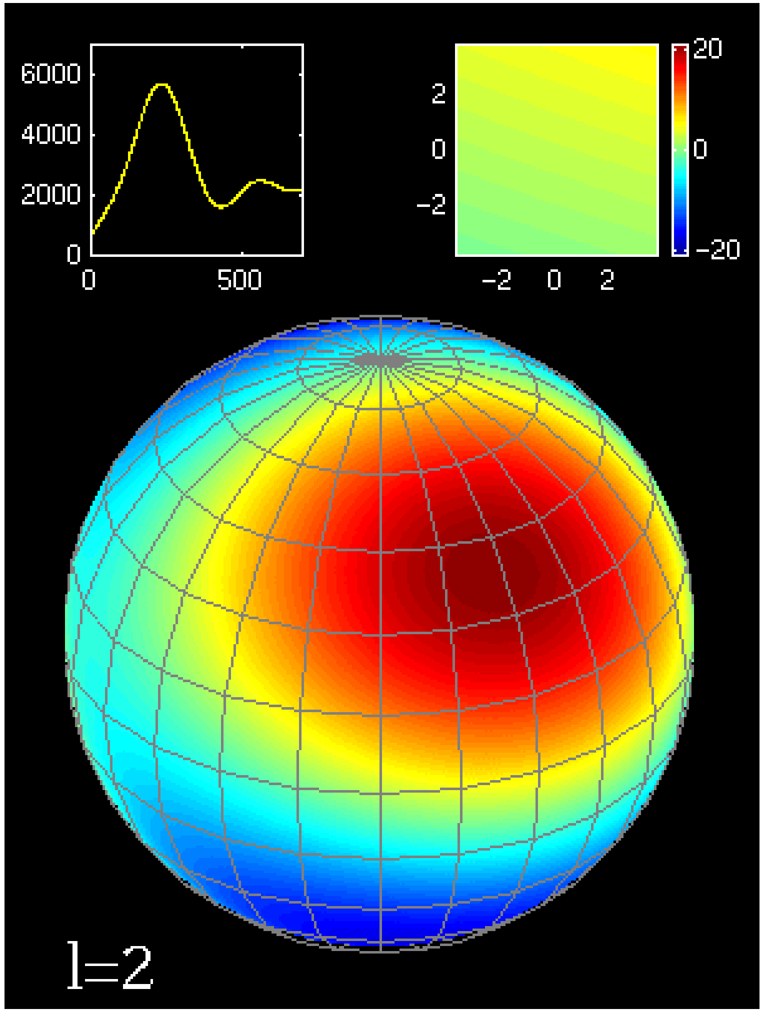


Cumulative image of the WMAP sky as incresing l numbers are summed

*(For each l, all orders m are accumulated)*

find.spa.umn.edu/~pryke/logbook/20000922/

# Application to wide-field imagery : WMAP images



Cumulative image of the WMAP sky as incresing l numbers are summed

*(For each l, all orders m are accumulated)*

find.spa.umn.edu/~pryke/logbook/20000922/

# Essential ideas on Spherical Harmonic expansion

- ☛ Signal depending on of 2 angular spherical coord. $(\theta,\phi)$
- ☛ Can be expressed as a series of spherical Harmonics $Y^l_m$ (base functions)
- ☛ $Y^l_m$ are orthonormal, with respect to an appropriate scalar product
- ☛ Coefs. of the series calculated as a scalar product between the signal and each $Y^l_m$
- ☛ $Y^l_m$ are connected to Laplace differential Eq. in spherical coordinates

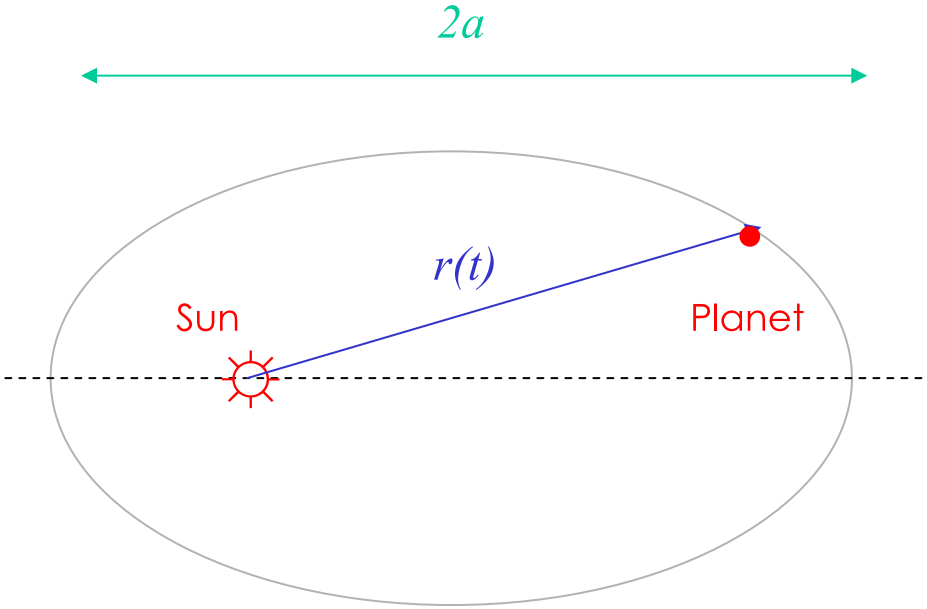
# Representation of signals as series of orthogonal functions

1. Fourier Series
2. Legendre polynomials
3. Spherical harmonics
4. Bessel functions

# Bessel, 1824 memoir

« Untersuchung des Theils der planetarischen Störungen, welcher aus der Bewegung der Sonne entsteht »

« Examination of that part of the planetary disturbances , which arises from the movement of the sun »



ε : excentricity
*T* : orbital period

Fourier series of the planet position *r(t)* :

$$\frac{r}{a} = 1 + \frac{\epsilon^2}{2} + \sum_{n=1}^{\infty} B_n \cos\left(2\pi \frac{nt}{T}\right)$$

«Bessel coefficient » :

$$B_n = -\frac{\epsilon}{n\pi} \int_0^{2\pi} \sin u \ \sin(nu - n\epsilon \sin u)\, du$$

# Bessel functions

Bessel's differential equation :

$$x^2 y'' + x y' + (x^2 - n^2) y = 0$$

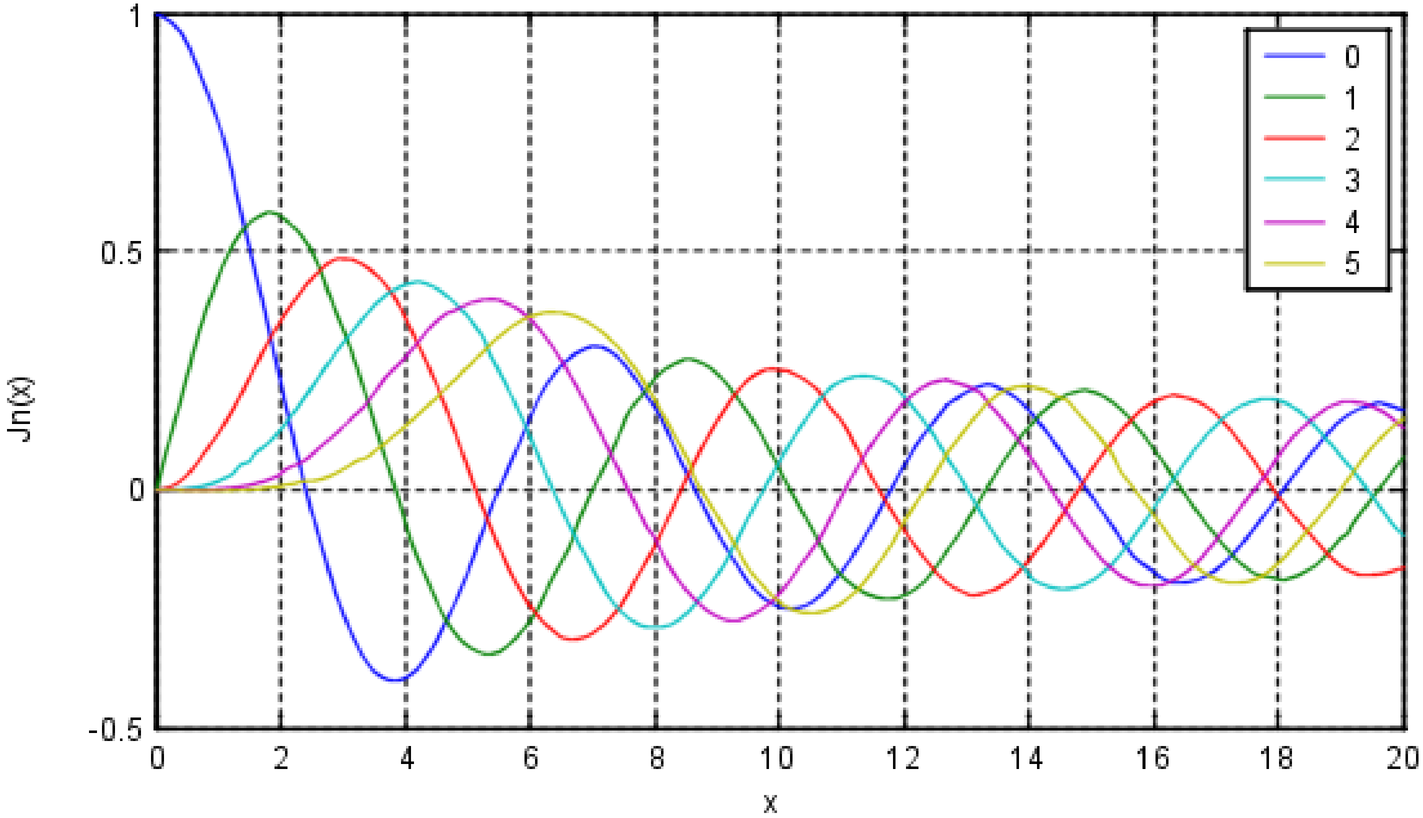
($n$ is a constant)
we consider $n$ integer here

Two base solutions : $J_n(x)$ and $Y_n(x)$

1st kind

2nd kind

# Bessel Functions of the 1st kind $J_n(x)$



Infinite number of roots

Regular for $x \to 0$ :

$J_n(0)=0$ for $n>0$ ; $J_0(0)=1$

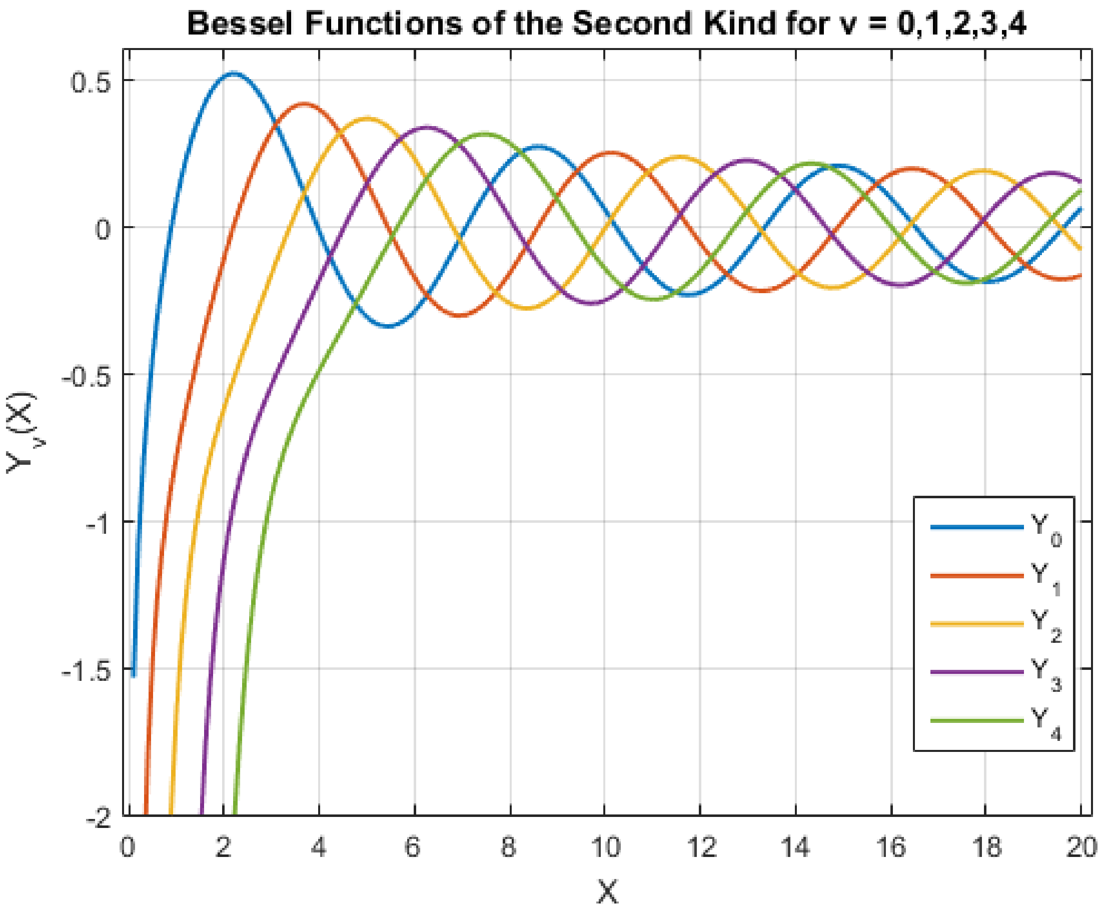
$$J_n(x) \simeq \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

Limit for $x \to \infty$ :

$$J_n(x) \simeq \sqrt{\frac{\pi}{2x}} \cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

~ damped sinudoid

# Bessel Functions of the 2nd kind $Y_n(x)$



Diverge at $x \to 0$ :

$$Y_n(x) \simeq -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n$$

$$Y_0(x) \simeq \frac{2}{\pi}\ln\left(\frac{x}{2}\right)$$

Limit for $x \to \infty$ :

$$Y_n(x) \simeq \sqrt{\frac{\pi}{2x}}\sin\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

damped sinudoid, phase-shifted with $J_n$

# Bessel Functions of the 1st kind $J_n(x)$

(with $n$ integer)

Series representation ($n>0$) :

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n}$$

$$J_{-n} = (-1)^n J_n$$

Parity : $n$

$J_n$ are the Fourier coefficients of the development :

$$e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\phi}$$

➡ « sort of » generating function as defined for legendre polynomials :

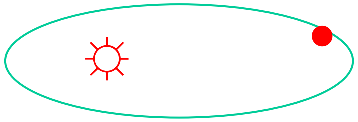
$$[1-2rx+r^2]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} r^n\, P_n(x)$$

with

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\phi}\, e^{-in\phi}\, d\phi$$

Integral representation for $J_n(x)$

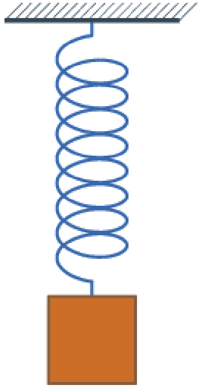
*Compare with Bessel coefficient*



$$B_n = -\frac{\epsilon}{n\pi}\int_0^{2\pi} \sin u\ \sin(nu - n\epsilon\sin u)\, du$$

# Bessel functions in physics

## 1 dimension : harmonic oscillator



Differential equation :

$$f'' + \omega^2 f = 0$$

$f(t)$ : spring extension

Base solutions :

$$e^{\pm i\omega t}$$

## 2 dimensions : membrane of a drum

$$\Delta f + k^2 f = 0$$

$f(\rho,\phi)$ : membrane elevation

(temporal dependence in $e^{-i\omega t}$)

Base solutions :

$$f_{nm}(\rho,\phi) = J_n(k_m\rho)\, e^{in\phi}$$

« mode »

$(\rho,\phi)$ : polar coordinates
$(m,n)$ : integers

General solution :
linear combination of modes $f_{mn}(\rho,\phi)$

Bessel function generally appear in polar (2D) or cylindrical (3D) coordinates

# Series expansions involving Bessel functions :

Neumann series

$$f(x) = \sum_{n=0}^{\infty} a_n \, J_n(x)$$

(can be generalized to real indexes $J_\nu$)

Convenient for expansions on $[-\infty, +\infty]$

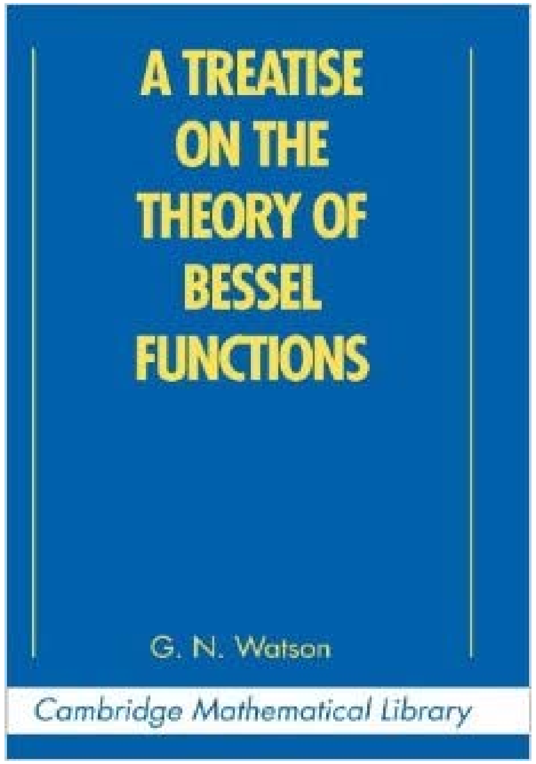
---

Fourier-Bessel series

$$f(x) = \sum_{n=1}^{\infty} c_n \, J_p(\alpha_{np} x)$$

$\alpha_{np}$ is the $n^{th}$ positive root of $J_p$

A reference book about
Bessel functions (800 pages)



A TREATISE
ON THE
THEORY OF
BESSEL
FUNCTIONS

G. N. Watson

Cambridge Mathematical Library

Convenient for expansions on $[0,1]$
with boundary condition at $x=1$

# Neumann series

Example :

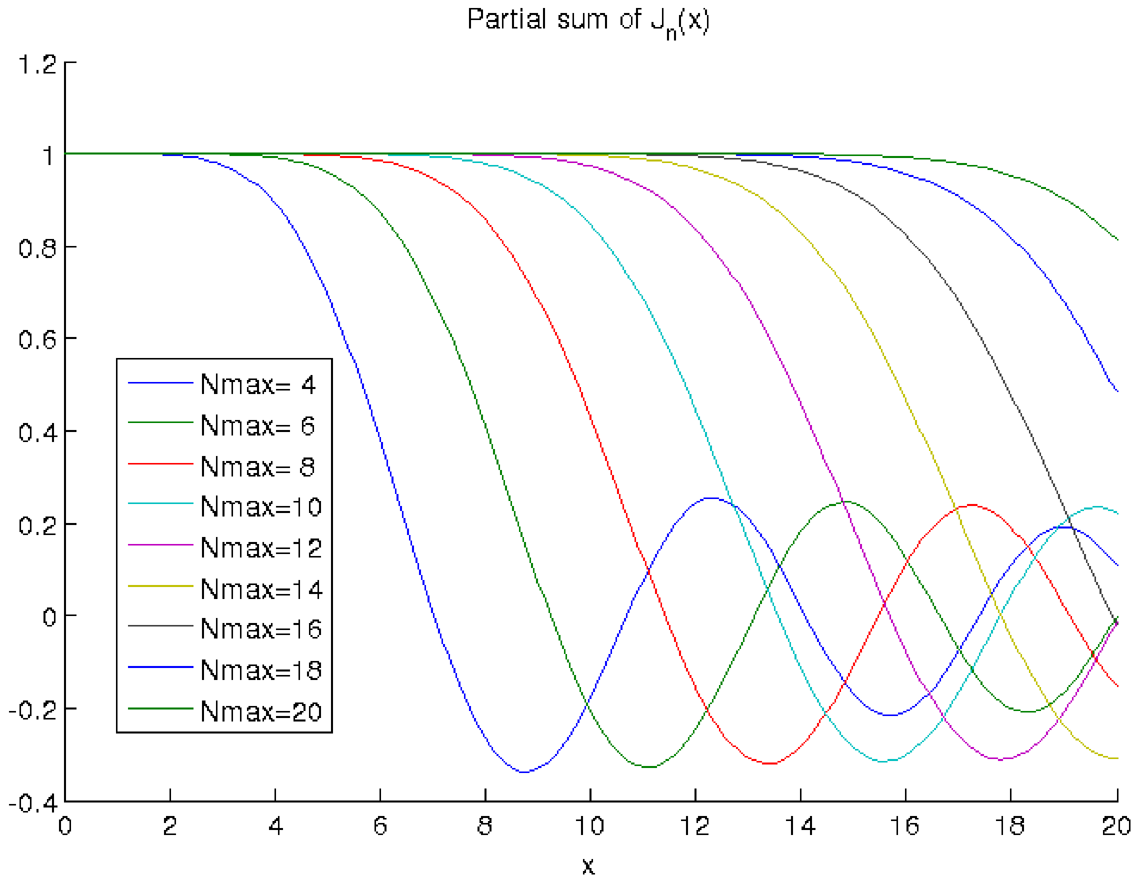
$$e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)\, e^{in\phi}$$

Fourier series for $\phi$ $(x=Cte)$

Neumann series for $x$ $(\phi=Cte)$

If $\phi=0$

$$1 = \sum_{n=-\infty}^{\infty} J_n(x)$$



# Drum modes and Fourier-Bessel series

$R$ : radius of the drum

Base solutions :

$$f_{mn}(\rho,\phi) = J_n(k\rho)\, e^{im\phi}$$

Must satisfy $f_{mn}(R)=0$ □$\phi$ (boundary condition) :

➡ $kR$ is a root of $J_n$

General solution is :

$$f(\rho,\phi) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} A_{nm} \boxed{J_n\left(\alpha_{nm}\frac{r}{R}\right)} \boxed{e^{in\phi}}$$

$\alpha_{nm}$ is the $m^{th}$ root of $J_n$

Fourier-Bessel expansion for $x=r/R$

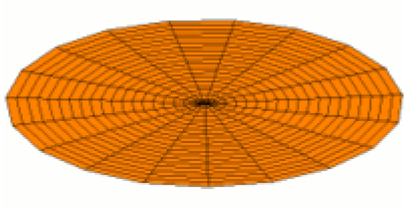
Fourier series for $\phi$

Single term $(n, m)$ : «mode»

# Drum vibration modes

$n=0$ (isotrope)

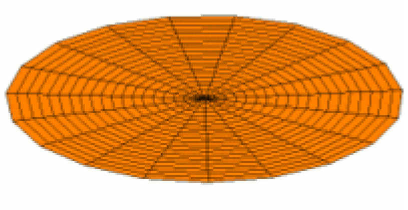
$m=1$



1st zero of $J_0$

$A_{01}\, J_0(\alpha_{01}\, r/R)$

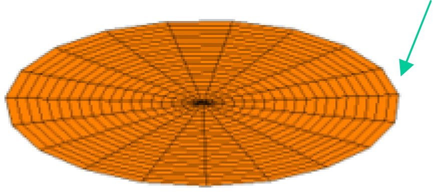
$m=2$



2nd zero of $J_0$

$A_{02}\, J_0(\alpha_{02}\, r/R)$

$m=3$

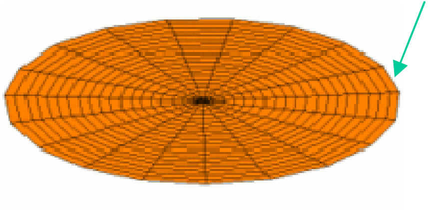


3rd zero of $J_0$

$A_{03}\, J_0(\alpha_{03}\, r/R)$

$n=1$ *(real part)*

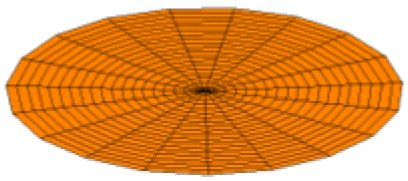
$m=1$



1st zero of $J_1$

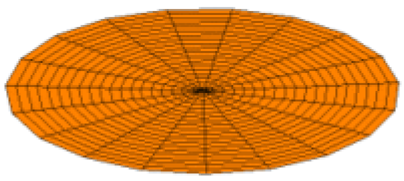
$A_{11}\, J_1(\alpha_{11}\, r/R) \cos \phi$

$m=2$



$A_{12}\, J_1(\alpha_{12}\, r/R) \cos \phi$

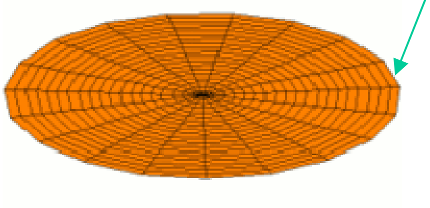
$m=3$



$A_{13}\, J_1(\alpha_{13}\, r/R) \cos \phi$

$n=2$ *(real part)*

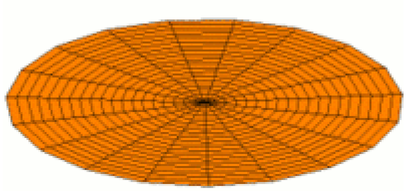
$m=1$



1st zero of $J_2$

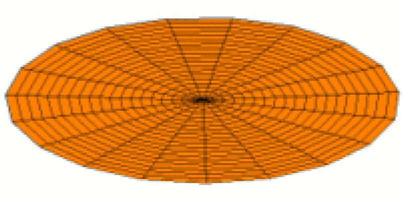
$A_{21}\, J_2(\alpha_{21}\, r/R) \cos 2\phi$

$m=2$



$A_{22}\, J_2(\alpha_{22}\, r/R) \cos 2\phi$

$m=3$



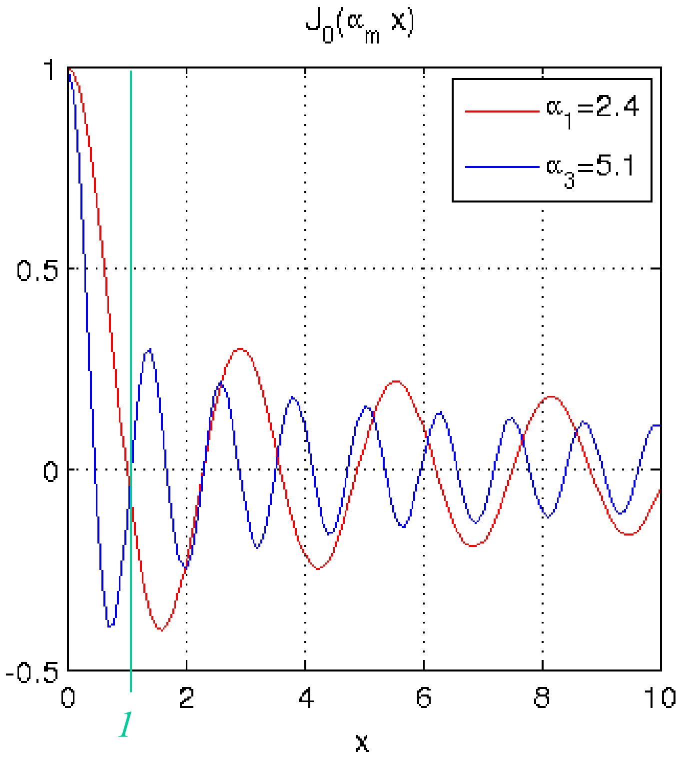
$A_{23}\, J_2(\alpha_{23}\, r/R) \cos 2\phi$

Source Wikipedia

# Orthogonality and Fourier-Bessel expansion

Scalar product :
$$(f,g) = \int_0^1 x\, f(x)\, g(x)\, dx$$

Orthogonality :
$$(J_n(\alpha_m x)\, J_n(\alpha_p x)) = \int_0^1 x\, J_n(\alpha_m x)\, J_n(\alpha_p x)\, dx = \frac{J_{n+1}^2(\alpha_m)}{2}\, \delta_{mp}$$

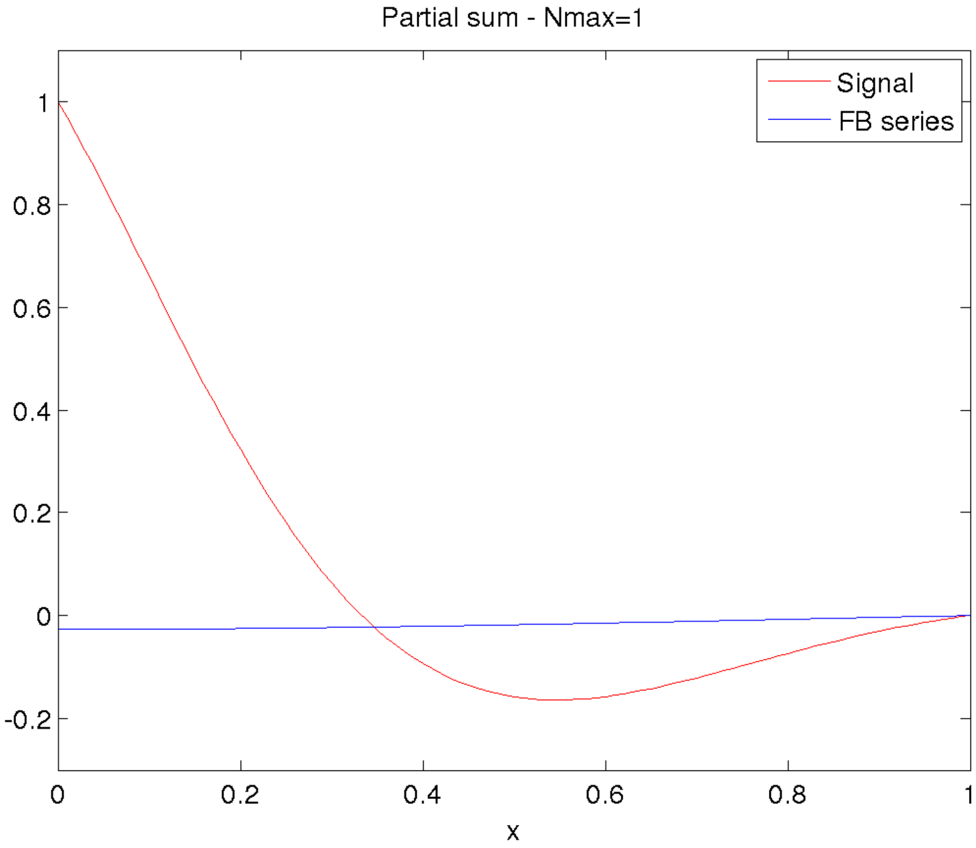


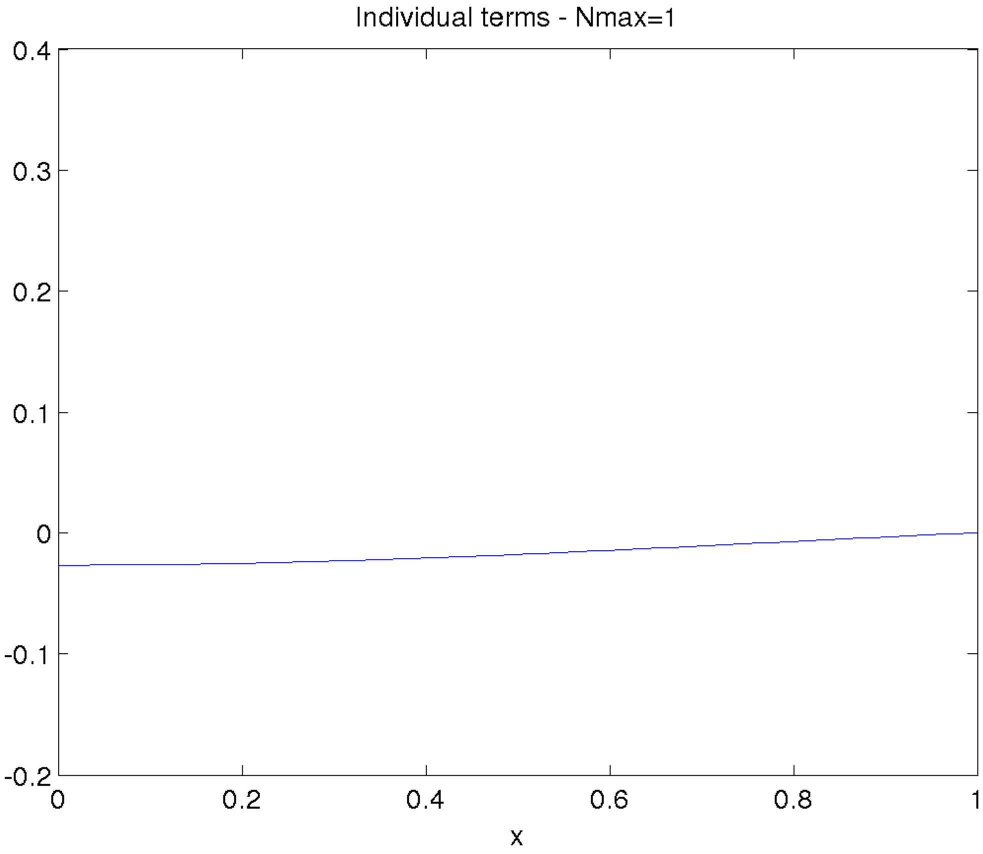
Fourier-Bessel series (in the interval [0,1]) :

$$\boxed{f(x) = \sum_{m=1}^{\infty} c_m\, J_n(\alpha_m x)}$$

Coefficient determination :
$$c_m = \frac{2}{J_{n+1}^2(\alpha_m)} \int_0^1 x\, f(x)\, J_n(\alpha_m x)\, dx$$

# Example of Fourier-Bessel reconstruction





Signal : $f(x) = e^{-3x} \cos\left(\frac{3\pi}{2} x\right)$

(null at $x=1$)

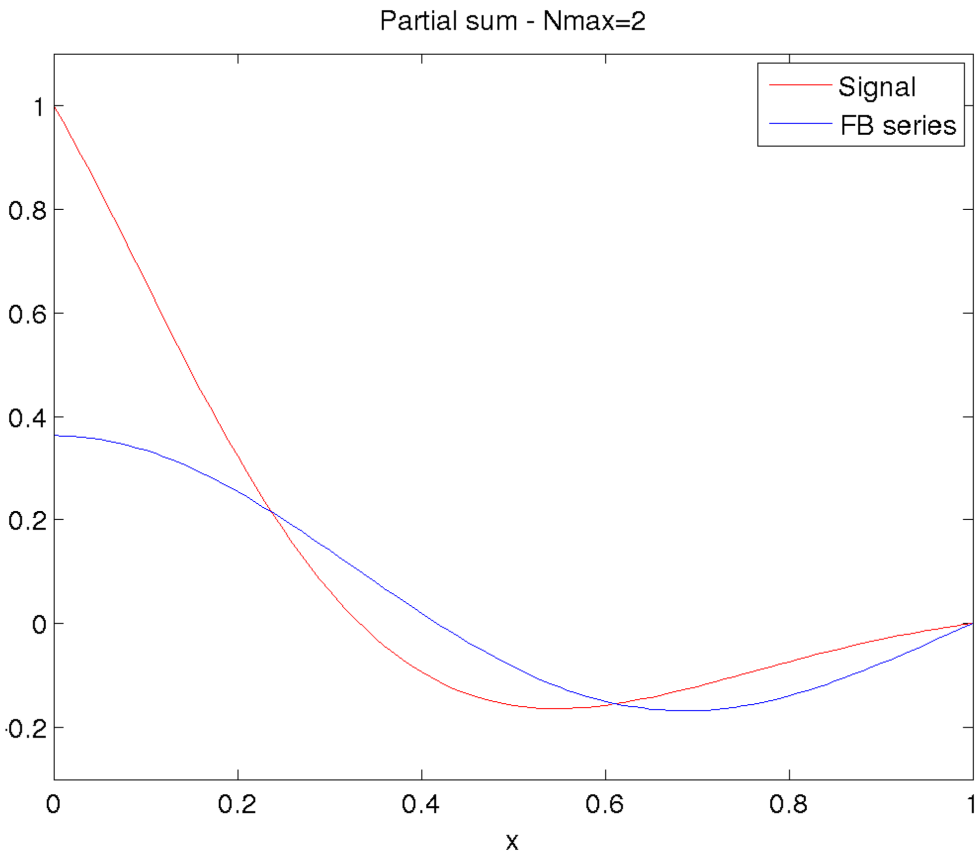
Expansion on $J_0$'s : $\sum_{m=1}^{N_{max}} c_m J_0(\alpha_m x)$

First term $m=1$

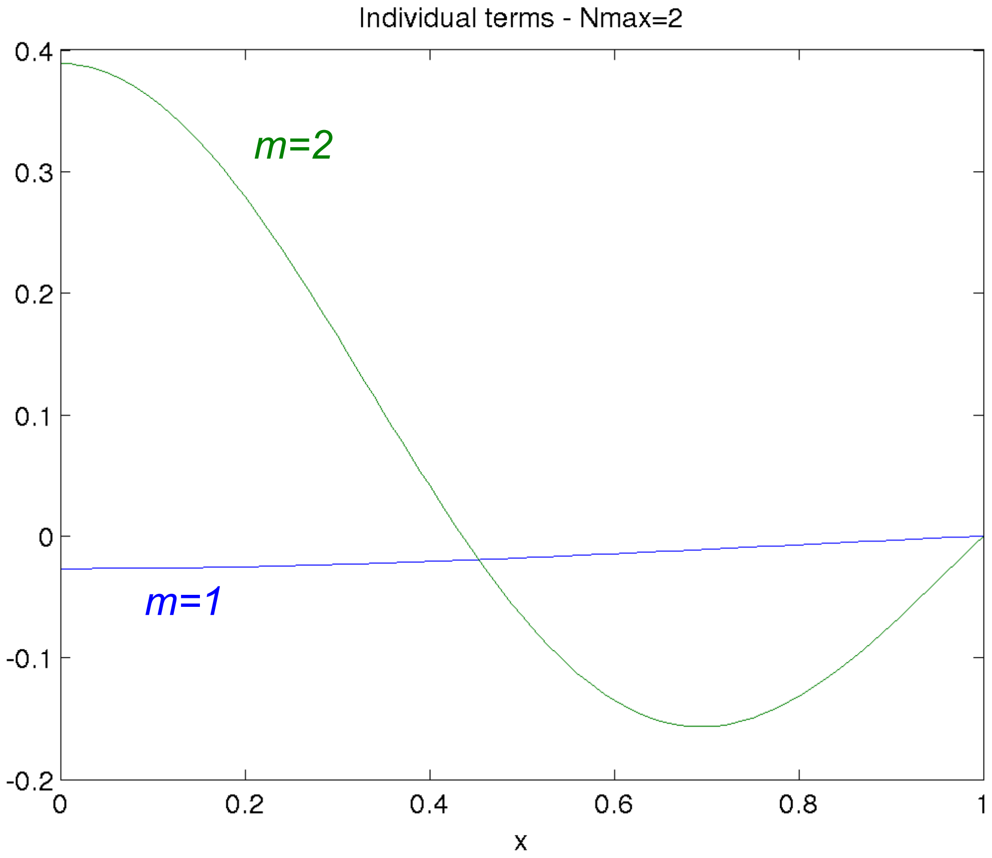
$c_1 J_0(\alpha_1 x)$

$\alpha_1=2.4048$

$c_1=-0.027$



Partial sum - Nmax=2



Individual terms - Nmax=2

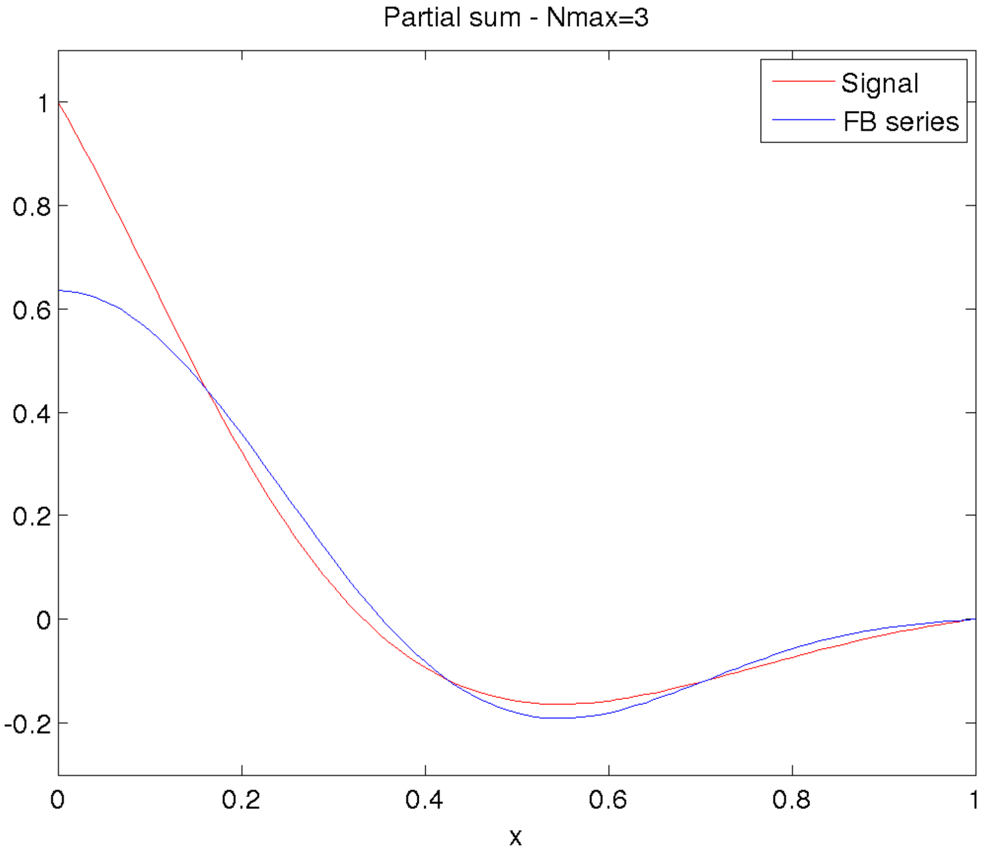
Signal : $f(x) = e^{-3x} \cos\left(\frac{3\pi}{2}x\right)$

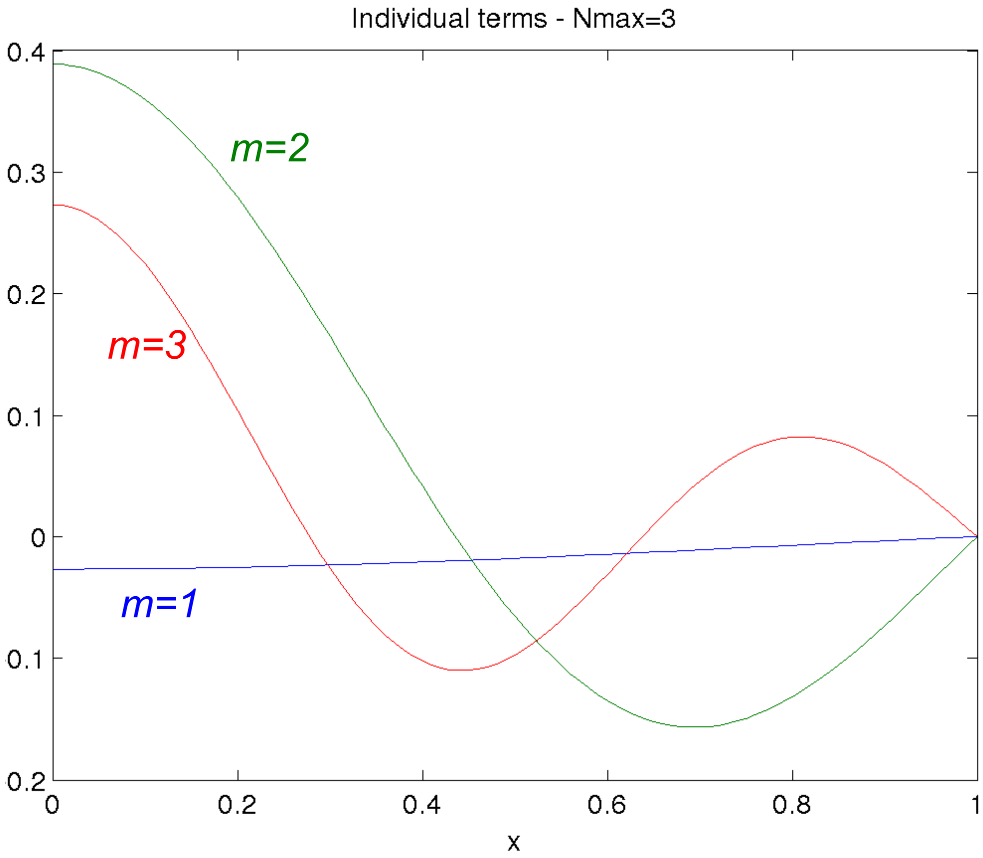
Individual terms $m=1,2$

$\alpha_1=2.4048$ $\alpha_2=5.5201$

$c_1=-0.027$ $c_2=0.389$

Expansion on $J_0$'s : $\sum_{m=1}^{N_{max}} c_m \, J_0(\alpha_m x)$

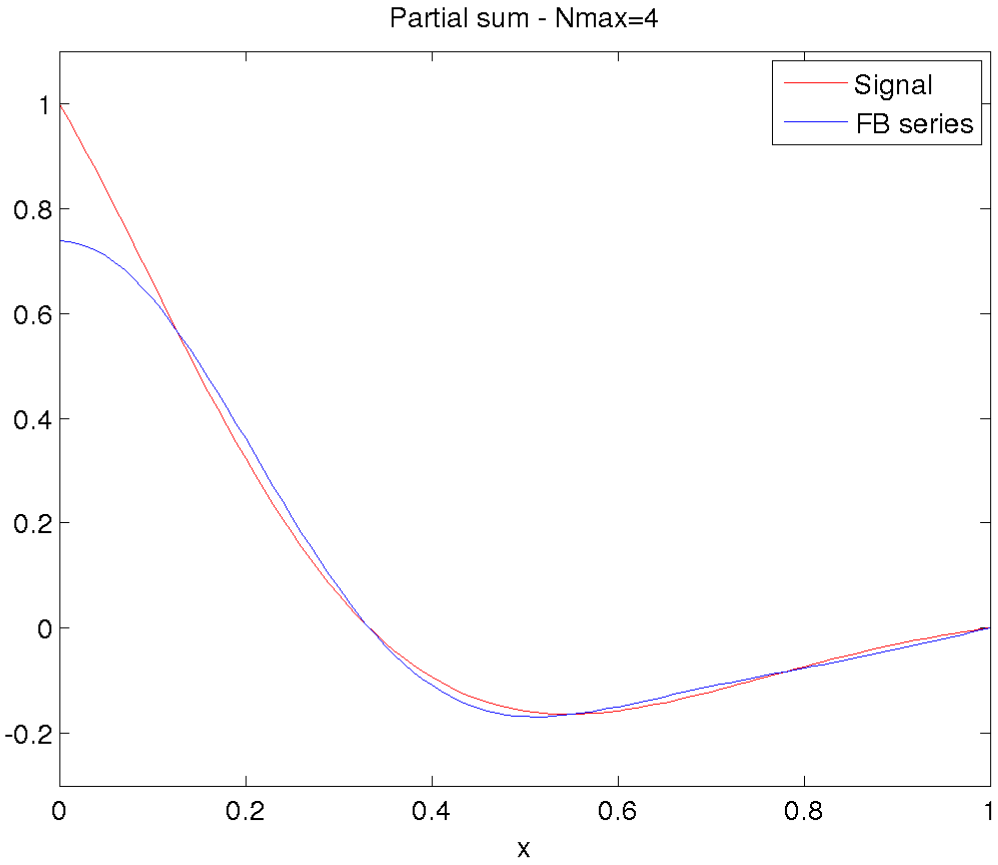




Signal : $$f(x) = e^{-3x}\cos\left(\frac{3\pi}{2}x\right)$$

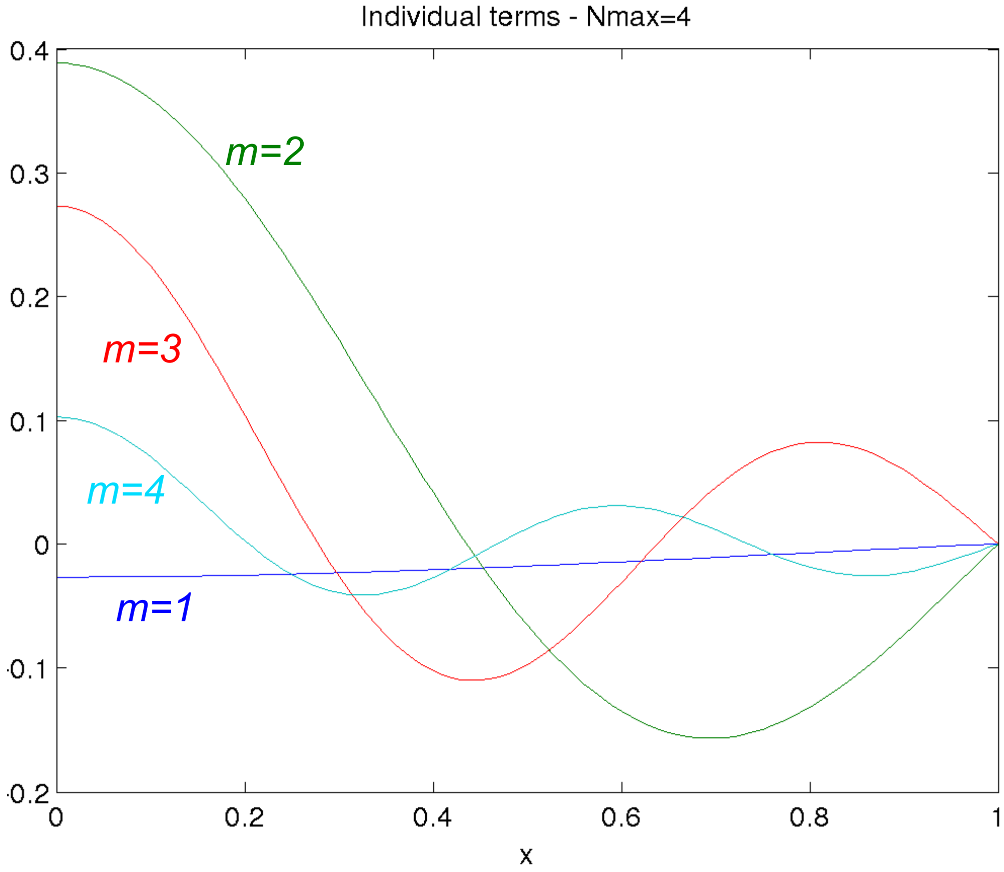
Individual terms *m=1,2,3*

Expansion on $J_0$'s : $$\sum_{m=1}^{N_{max}} c_m\, J_0(\alpha_m x)$$

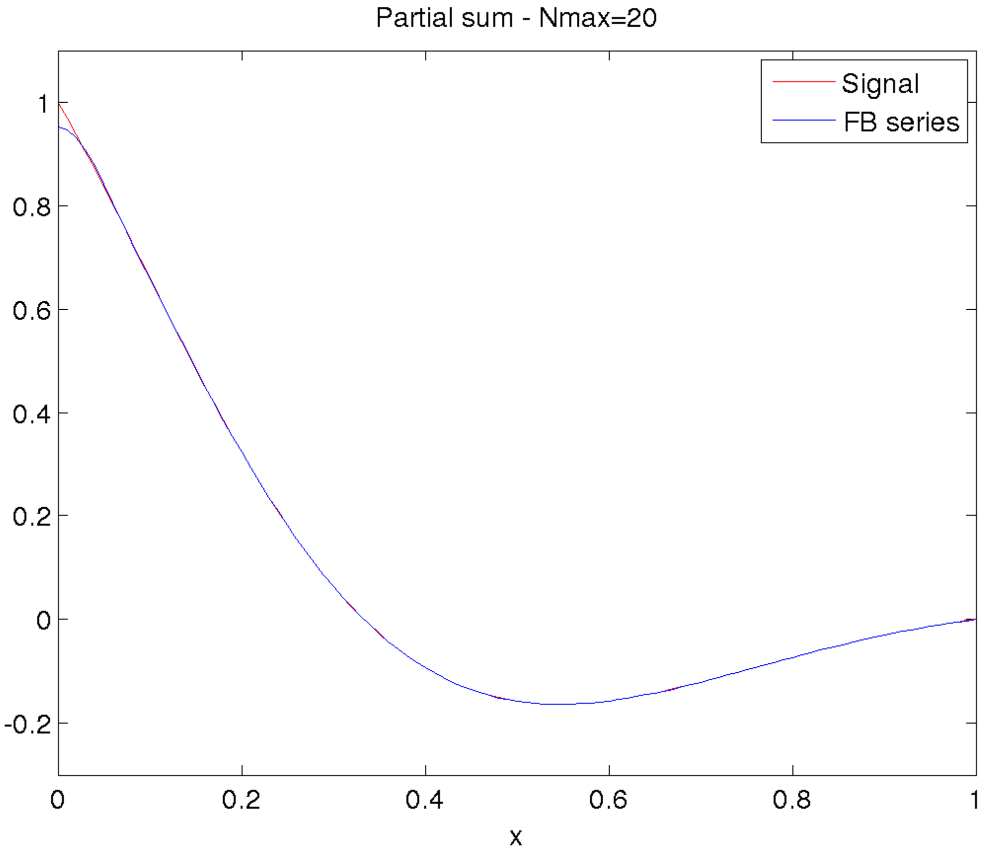


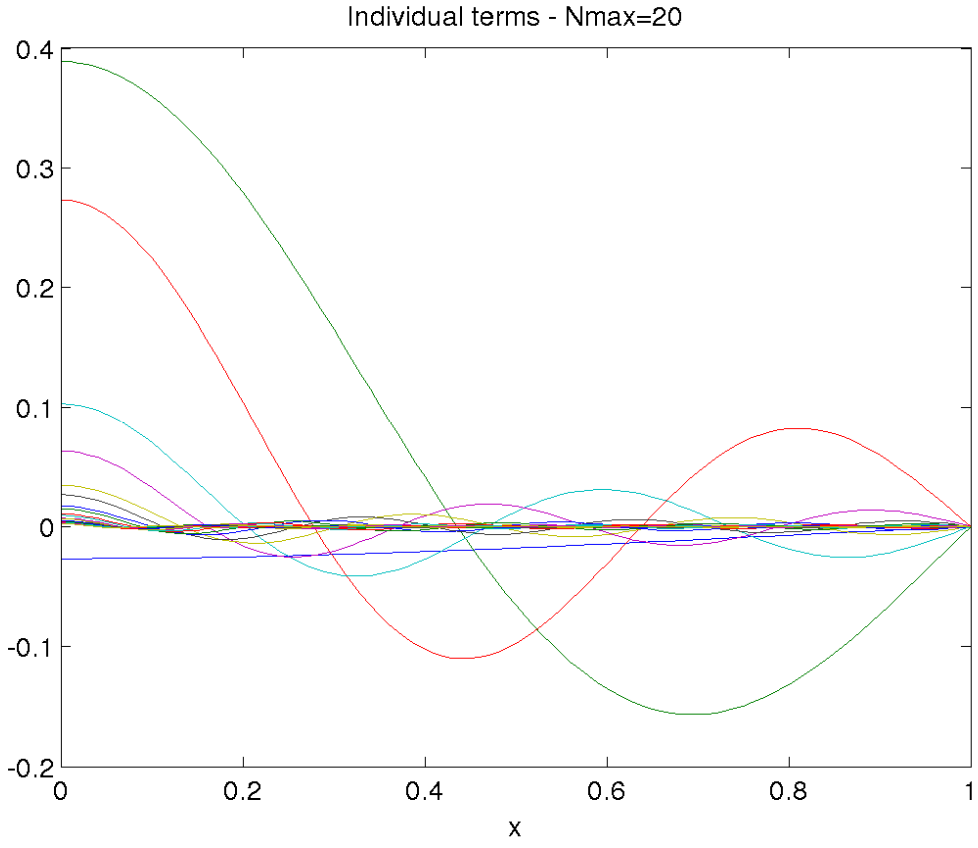
Signal : $f(x) = e^{-3x} \cos\left(\frac{3\pi}{2} x\right)$

Expansion on $J_0$'s : $\sum_{m=1}^{N_{max}} c_m J_0(\alpha_m x)$



Individual terms *m=1,2,3,4*



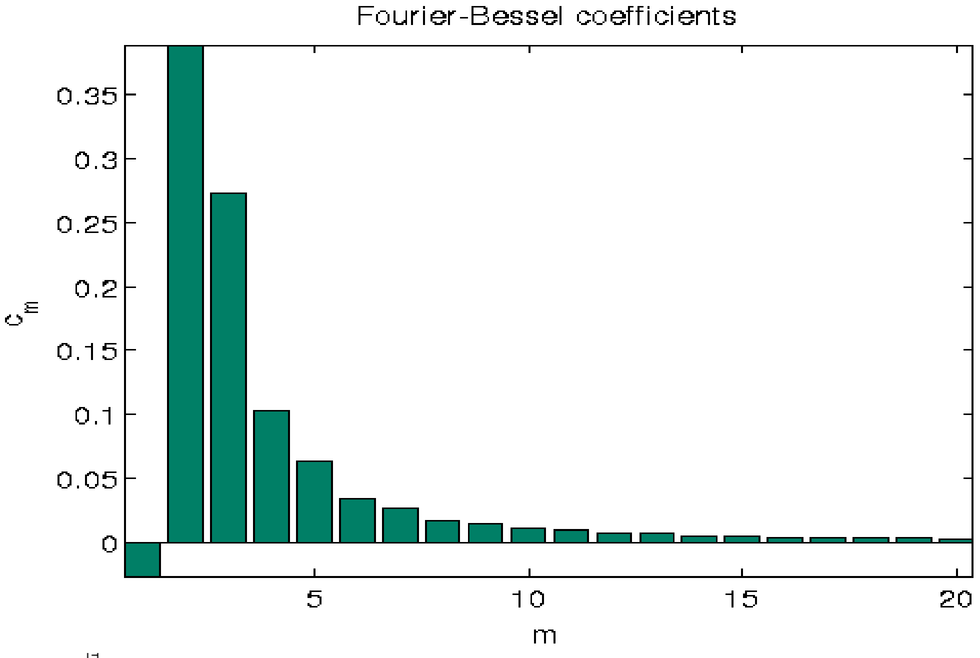


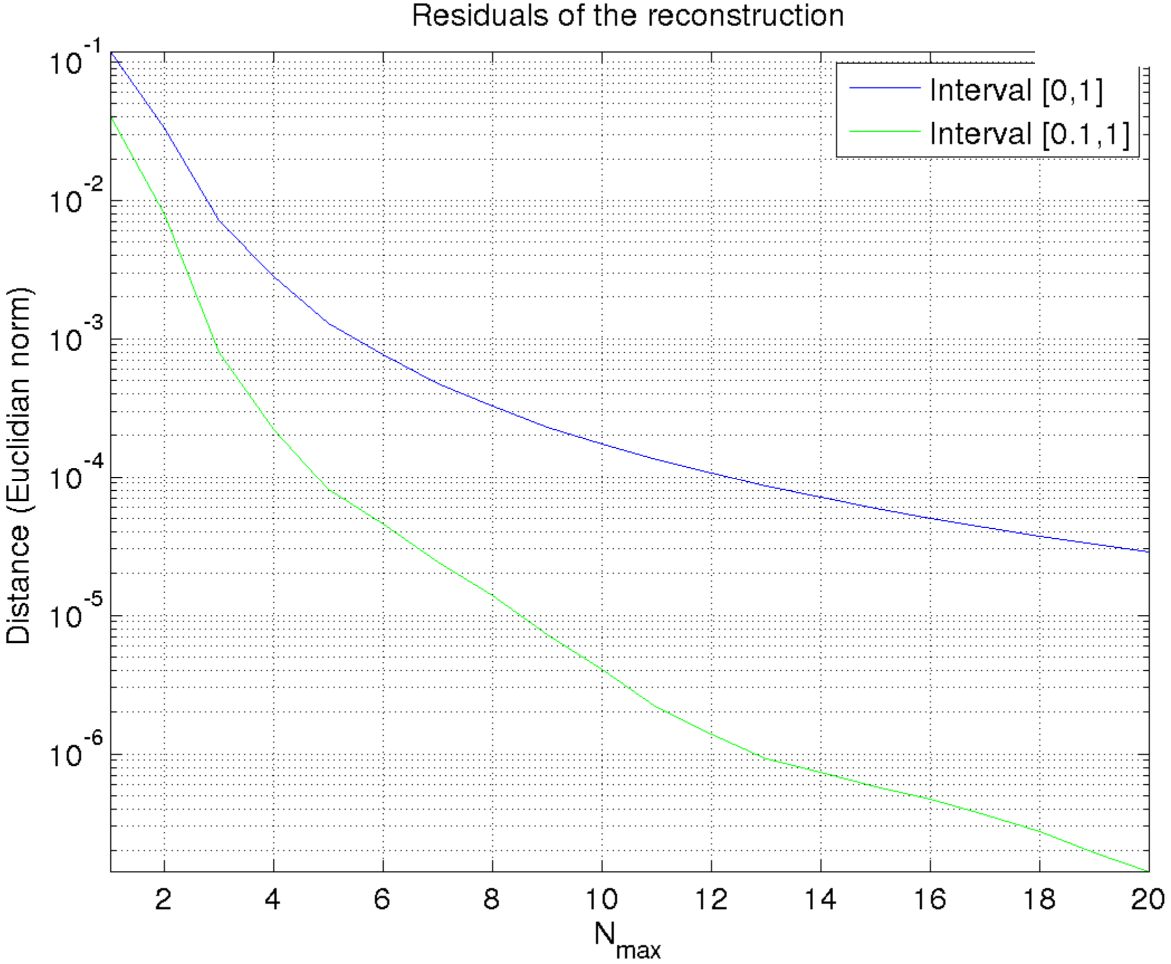
Signal : $f(x) = e^{-3x} \cos\left(\frac{3\pi}{2}x\right)$

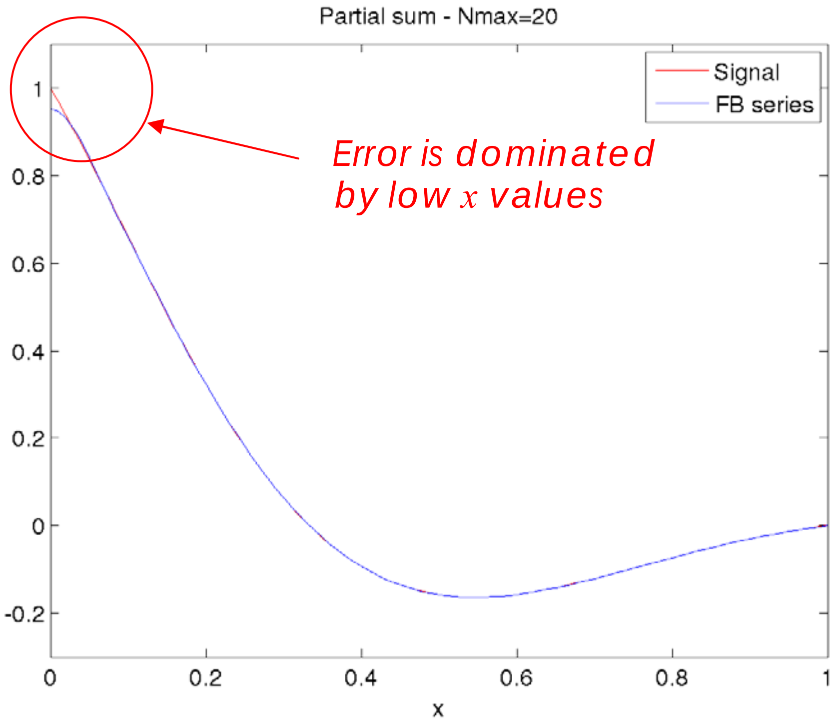
Individual terms $m=1$ to $20$

Expansion on $J_0$'s : $\sum_{m=1}^{N_{max}} c_m J_0(\alpha_m x)$

# Results of the reconstruction


Fourier-Bessel coefficients


Residuals of the reconstruction


Partial sum - Nmax=20

*Error is dominated by low $x$ values*

## Essential ideas :

- ☛ Signal or function defined on an interval
- ☛ Can be expressed as a sum of orthogonal base functions
- ☛ Need for a scalar product (depends on the base functions)
- ☛ Base functions defined by a differential Eq. or a characteristic function
- ☛ Choice of the base : free or induced by the physics